THE BUNDLE OF A LATTICE GAUGE FIELD

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ABSTRACT. Given a smooth manifold M and a Lie group G, we consider parallel transport maps –groupoid homomorphisms from a path groupoid in M to G– as an alternative description of principal G-bundles with smooth connections on them. Using a cellular decomposition $\mathscr{C}$ of M, and a system of paths associated to $\mathscr{C}$, we define a homotopical equivalence relation of parallel transport maps, leading to the concept of an extended lattice gauge (ELG) field. A lattice gauge field, as used in Lattice Gauge Theory, is part of the data contained in an ELG field, but the latter contains additional topological information of local nature, sufficient to reconstruct a principal G-bundle up to equivalence, in the spirit of Barrett [3]. Following a theorem of Pachner [15], we give a criterion to determine when two ELG fields over different cell decompositions $\mathscr{C}$ and $\mathscr{C}'$ define equivalent bundles. As a first concrete physical application, we define a simple operation that captures the geometric essence of the 't Hooft loop operator in the theory of quantum gauge fields, and study its relation with the bundle structure of an ELG field.

CONTENTS

1. INTRODUCTION

The main objective of this paper is to study how the topology of principal bundles may be incorporated into the formulation of Lattice Gauge Theory. The standard geometric approach to the study of field theories with gauge

symmetry, such as the Yang-Mills theory, considers a principal G-bundle $P \to M$ over a smooth manifold M, from which a space of "states" $\mathscr{A}/\mathscr{G}$, consisting of smooth connections in P, modulo the action of the group of gauge transformations, is induced. However, in the standard formulation of Lattice Gauge Theory on $\mathbb{R}^n$ (or more generally, $\mathbb{R}^n/\Lambda$ for a lattice Λ) [5], the discrete analog of a gauge field, the so-called *lattice gauge field*, corresponds geometrically to the evaluation of the parallel transport of a connection (a gauge-invariant notion) over a discrete collection of paths, generated by a lattice. Our motivation is the observation that, while such notion captures the essence of a discrete gauge field (a question of a local nature), it is insufficient to characterize the topology of P: the information on how to glue together local trivializations is missing. While the question of construction of topological charges in special cases and in reduced regimes is known (e.g. [14], where a physical application is given in the case dim $M = 2$, $G = \mathrm{SO}(2)$, and [16], where the focus is in the non-abelian case dim $M = 4$, $G = \mathrm{SU}(2)$), it is our impression that the previous problem has not yet been explored comprehensively in the existent literature.

In the continuum, the Barrett-Kobayashi construction [3] ensures that a suitable axiomatization of the notion of *holonomy* of a connection [9, 1], which is based on the structure of the space of based loops in M modulo thin equivalence, is sufficient to reconstruct a pair (P, A), consisting of a principal G-bundle and a smooth connection on it. Since a lattice gauge field may be conceived as a discretization of a smooth holonomy map, It is then natural to infer, from the very nature of the Barrett-Kobayashi construction, that a refined notion of discrete gauge field, which we call an *extended lattice gauge field*, can be given instead, as an equivalence class of gauge fields, under a suitable topological equivalence relation, in such a way that the topology of a principal G-bundle prevails through the decimation process. Such is the basic idea that we exploit.

The general problem of classification of isomorphism classes of principal G-bundles over M, has a natural homotopical interpretation [13] in terms of the classifying space BG, and it is ultimately related to the structure of the homotopy spaces $\pi_1(G, e), \ldots, \pi_{n-1}(G, e)$. The Čech cohomology with coefficients in G describes the classification as well, and for the n-sphere, $n \geq 2$, the answer to the classification problem is greatly simplified, since there is a correspondence between equivalence classes of smooth principal G-bundles $P \to S^n$, and isotopy classes of smooth functions $g : S^{n-1} \to G$, the *clutching* maps for P. Part of our strategy is to take advantage of the fundamental principle of local triviality to the understanding of an arbitrary principal G-bundle $P \to M$. We provide an alternative homotopical interpretation of an equivalence class of smooth principal G-bundles, which has the advantage of being constructive, more explicit, and more amenable for computations than the standard one. The essence of the problem is translated into a combinatorial one. The starting point will consist on choosing an auxiliary "scaffolding" structure on M, namely a special but generic type

of cellular decomposition $\mathscr{C}$, which is dual to a triangulation in M, together with a network of paths Γ joining a series of based points for every pair of neighboring cells. Then we adapt the notion of an equivalence class of Čech cocycles, to a homotopy class of collections of compatible clutching maps. The intrinsic triangulation then serves the practical purpose of organizing the new homotopy data, as the clutching maps we consider are defined over the $(n-1)$-skeleton of $\mathscr{C}$, and the compatibility conditions they satisfy are systematically encoded over the $(n-2)$-skeleton of $\mathscr{C}$.

Our first important result, theorem 1, is a reconstruction theorem, in the spirit of Barrett [3], adapted to the notion of parallel transport of local path families. Equipped with it, we introduce the notion of an extended lattice gauge field. Then, we proceed to dissect the extended lattice fields, into elementary local pieces (theorem 2). Such pieces consist of the standard LGT data, namely, the values of parallel transport over a network of paths in M, but also contain homotopy classes of extensions of glueing maps from the boundary of a $(k+1)$-cell –a k-sphere– to its interior. In the process, we are able to identify the minimal local topological data, contained in an extended lattice gauge field, that is sufficient to reconstruct a given bundle, which we name the *core* of an extended lattice gauge field. Such a glossary of local data turns out to reveal a *torsor* structure in the space of extended lattice gauge fields for a given triple $(M, \mathscr{C}, G)$, extending the group structure on the space of standard LGT data. Moreover, the natural fibration from the space of extended lattice gauge fields onto the space of isomorphism classes of principal G-bundles, reveals a new generalized homogeneous space structure in the latter. While not the main objective our our present investigation, the characterization of the space of isomorphism classes of principal G-bundles as a generalized homogeneous space, which is inspired by the holonomy point of view, provides a new promising structure, of combinatorial nature, to organize and classify its internal structure. In particular, it should be possible to reconstruct the Chern-Weil theory of characteristic classes from such a combinatorial point of view. Such questions deserve to be investigated separately, and we are planning to address them in the near future.

We have described results which build on the introduction of an additional structure over a manifold M, namely a triangle-dual cellular decomposition. Our results concerning principal bundles over M can be freed from such an auxiliary structure. We use a dual version of the fundamental theorem of U. Pachner [15], relating any two triangulations of M by a series of elementary transformations, to provide a criterion that determines when two extended lattice gauge fields, defined over different cellular decompositions, define equivalent principal G-bundles (theorem 3). It is in that way that our results are freed from their previous dependence on a cellular decomposition.

We complement the previous results with a concrete physical application. We define a simple geometric operation on a gauge field, that captures the geometric essence of the 't Hooft loop operator in the theory of quantum gauge fields. In theorem 4, We study its effect in the bundle structure of

the extended lattice gauge fields. Interestingly, our approach leads to the notion of a non-abelian gerbe, as the natural geometric framework for the study the 't Hooft operation.

The work is organized as follows. In section 2, we introduce the special cellular decompositions we will work with. Section 3 is a short introduction to path structures and local path families, which are exploited in section 4, in order to introduce the cellular parallel transport maps, the reconstruction theorem for bundles with connection, and the extended lattice gauge fields. Section 5 constitutes the heart of the article, where the relation between extended lattice gauge fields and isomorphism classes of principal bundles is studied, and is complemented with section 6, describing explicitly the extended lattice gauge fields in small dimensions. Section 7 describes the dependence of the bundle structure of the extended lattice gauge fields on different cellular decompositions, while section 8 concludes with a study of the 't Hooft loop operation in the present topological context. In addition, two appendices complement our work. Appendix A is a brief introduction to cellular decompositions, where the genericity of the triangle-dual ones is justified, while appendix B contains a proof of the correspondence between the space of equivalence classes of principal G-bundles, and the space of equivalence classes of local homotopy data in the core of an ELG field.

2. Triangle-dual cellular decompositions

Let M be an n-dimensional smooth manifold. It is a classical result of J. H. C. Whitehead that M admits a piecewise-linear (P.L.) structure, namely, a triangulation Δ for which the link of any simplex is a piecewise-linear sphere, and moreover, any two such P.L. structures are related by a piecewise-smooth bijection. It would be sufficient to assume the piecewise-smoothness of any such Δ. For any triangulation Δ, we will denote its 0-simplices by $v, w, \ldots$ and unless otherwise stated, a given 1-simplex by τ. The letter σ will denote an arbitrary k-simplex. We will assume henceforth that M is orientable, and that a choice of orientation has been given to it.

Definition 1. A cellular decomposition $\mathscr{C} = \sqcup_{k=0}^{n} \mathscr{C}_k$ of M is called *triangle-dual* if there exist an open cover $\mathfrak{U} = \{\mathscr{U}_v\}_{c_v \in \mathscr{C}_n}$ of M with the following properties:

  (i) For every $\sigma \in N(\mathfrak{U})$ (the nerve of $\mathfrak{U}$, an abstract simplicial complex), the open set $\mathscr{U}_\sigma = \cap_{v \subset \sigma} \mathscr{U}_v$ is contractible,
  (ii) The geometric realization of $N(\mathfrak{U})$ is a P.L structure for M. In particular, $N(\mathfrak{U})$ is pure and $\cap_{i=0}^{n+1} \mathscr{U}_{v_i} = \emptyset$ for all pairwise-different $c_{v_0}, \ldots, c_{v_{n+1}} \in \mathscr{C}_n$,
  (iii) There is a 1-to-1 correspondence between the k-simplices $\sigma \in N(\mathfrak{U})$ and the $(n-k)$-cells $c_\sigma \in \mathscr{C}_{n-k}$, in such a way that $\overline{c_\sigma} \subset \mathscr{U}_\sigma$.

Remark 1. The motivation behind this definition is that, for any choice of P.L. structure Δ of M, the dual cellular decomposition $\Delta^\vee$ of M, which

is constructed in terms of the canonical barycentric subdivision of Δ, is triangle-dual with respect to the open cover of M given by the stars of the vertices of Δ (figure 1). Conversely, given a triangle-dual cellular decomposition $\mathscr{C}$ of M, it is also possible to provide a full characterization of a P.L. structure Δ such that $\Delta^\vee = \mathscr{C}$ if we are allowed to strengthen the choice of the cover $\mathfrak{U}$. Namely, by a *star-like cover* we mean an open cover where, moreover, $\mathscr{D}_k = \mathscr{S}_k - \mathscr{S}_{k-1}$, $k = 0, \dots, n$ is a disjoint union of embedded smooth k-disks, for $\mathscr{S}_{-1} = \emptyset$, and

$$\mathscr{S}_k = M \setminus \bigcup_{|\sigma| = k+2} \mathscr{U}_\sigma, \quad k = 0, \dots, n.$$

This way, the components of each $\mathscr{D}_k$ correspond to the interiors of the k-simplices of Δ, and $\mathscr{S}_k$ corresponds to the k-th skeleton of a triangulation Δ. Observe that, in particular, for any k-cell $c_\sigma \in \mathscr{C}_k$, there is an induced triangle-dual cellular decomposition of $\partial \overline{c_k}$ (a piecewise-smooth $(k-1)$-sphere in M).

To complement the previous picture, and justify the naturality of triangle-dual cellular decompositions, a genericity property satisfied by them is stated and proved in lemma 6 in appendix A.

FIGURE 1. A triangle-dual cellular decomposition $\mathscr{C}$ on a surface.
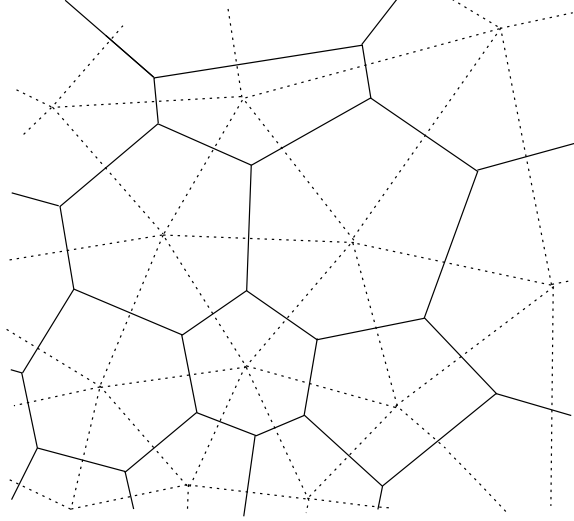
Remark 2. A very important combinatorial feature that distinguishes the triangle-dual cellular decompositions can be described as follows. When a cellular decomposition $\mathscr{C}$ is triangle-dual, and $c_\sigma \in \mathscr{C}_k$, there are exactly $n - k + 1$ $(k+1)$-cells $c_{\sigma_1}, \dots, c_{\sigma_{n-k+1}} \in \mathscr{C}_{k+1}$ such that $\cap_{j=1}^{n-k+1} \overline{c_{\sigma_j}} = \overline{c_\sigma}$. This is so since, over the dual abstract simplicial complex Δ, the $n - k + 1$ different $(n - k - 1)$-simplices $\sigma_1, \dots, \sigma_{n-k+1}$ determine the boundary $\delta(\sigma)$

of a unique $(n-k)$-simplex σ. If necessary, one could denote such c_σ by $c_{\sigma_1 \dots \sigma_{n-k+1}}$. Following the same logic, we conclude that for every $c_\sigma \in \mathscr{C}_k$, $\overline{c_\sigma}$ can be expressed as the intersection of exactly $n-k+1$ closures of n-cells $\overline{c_{v_1}}, \dots, \overline{c_{v_{n-k+1}}}$. As a consequence, one has the following. Whenever there is a triple $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\} \subset \mathscr{C}_{n-1}$ whose closures intersect nontrivially in $\overline{c_\sigma}$, for some $c_\sigma \in \mathscr{C}_{n-2}$, there is a unique triple $\{c_{v_1}, c_{v_2}, c_{v_3}\} \subset \mathscr{C}_n$ such that

$$(2.1) \qquad \overline{c_{\tau_1}} = \overline{c_{v_2}} \cap \overline{c_{v_3}}, \qquad \overline{c_{\tau_2}} = \overline{c_{v_3}} \cap \overline{c_{v_1}}, \qquad \overline{c_{\tau_3}} = \overline{c_{v_1}} \cap \overline{c_{v_2}},$$

and

$$(2.2) \qquad \overline{c_{\tau_1}} \cap \overline{c_{\tau_2}} \cap \overline{c_{\tau_3}} = \overline{c_{v_1}} \cap \overline{c_{v_2}} \cap \overline{c_{v_3}} = \overline{c_\sigma}.$$

Remark 3. Over a triangle-dual cellular decomposition, an orientation in M induces an orientation on the elements of each complete descending flag in $\mathscr{C}$. Once a k-cell c_τ has been given an orientation, there is an induced orientation on every $(k-1)$-cell $c_\sigma \subset \partial \overline{c_\tau} \cong S^{k-1}$. When the cellular decomposition is triangle-dual, the two different orientations that any k-cell c_σ may acquire can be described as follows. Every gapless descending flag ending in $\overline{c_\sigma}$ is of the form

$$\overline{c_{v_1}} \supset \overline{c_{v_1}} \cap \overline{c_{v_2}} \supset \cdots \supset \bigcap_{j=1}^{n-k+1} c_{v_j},$$

and moreover, any other gapless flag would correspond to a permutation of the ordered $(n-k+1)$-tuple $(c_{v_1}, \dots, c_{v_{n-k+1}})$. Then, two different gapless flags ending at the same element induce the same orientation if and only if they differ by an even permutation.[1] Moreover, for every ordered triple $\{c_{v_1}, c_{v_2}, c_{v_3}\}$ as in remark 2, we can decree an orientation in the induced triple $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\}$ from the choice of flags $c_{v_1} \supset \overline{c_{\tau_2}}$, $c_{v_2} \supset \overline{c_{\tau_3}}$ and $c_{v_3} \supset \overline{c_{\tau_1}}$, in such a way that the orientations furtherly induced in c_σ coincide. We will say that the triple

$$(2.3) \qquad \{\overline{c_{v_1}} \supset \overline{c_{\tau_2}} \supset \overline{c_\sigma}, \quad \overline{c_{v_2}} \supset \overline{c_{\tau_3}} \supset \overline{c_\sigma}, \quad \overline{c_{v_3}} \supset \overline{c_{\tau_1}} \supset \overline{c_\sigma}\}$$

has been given a *cyclic orientation*, which is clearly invariant under cyclic permutations of the indices. Given any $c_\sigma \in \mathscr{C}_{n-2}$, there are two choices of cyclic orientations in its corresponding triple of flags (2.3). Equivalently, one may reconstruct a cyclic orientation from a choice of orientation of the 2-simplex σ.

In the case when the manifold M is an Euclidean space or a quotient of it by a lattice (i.e., a cylinder or a torus), it would be of paramount importance to relate their standard cell decompositions generated by a square lattice in $\mathbb{R}^n$ to the triangle-dual cell decompositions. Hence, we include the following lemma, justifying that the constructions and results of this work are also relevant and applicable in those standard cases.

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[1] The previous definition is compatible with the corresponding notion of orientation of the simplex σ.

Lemma 1. *The cell decomposition $\mathscr{C}^0$ of $\mathbb{R}^n$ induced by the rectangular lattice $\Lambda^0 = \mathbb{Z}^n$ can be regarded as a degeneration of a family of triangle-dual cellular decompositions $\mathscr{C}^\epsilon$, $\epsilon \in (0, 1)$.*

Proof. Setting a 1-1 correspondence between n-cells c_v^0 in $\mathscr{C}^0$ and the 0-subcell $(i_1^v, \ldots, i_n^v)$ in their closure, where for each $k = 1, \ldots, n$,

$$i_k^v = \min\{i_k \,:\, (i_1, \ldots, i_n) \in \overline{c_v}\},$$

we can identify the n-cells in $\mathscr{C}^0$ with the elements in Λ^0. For any $\epsilon \in (0, 1)$, consider the lattice Λ^ϵ in $\mathbb{R}^n$, of n-tuples of real numbers of the form

$$(i_1, i_2 + \epsilon i_1, \ldots, i_{n-1} + \epsilon i_{n-2}, i_n + \epsilon i_{n-1}), \qquad i_1, \ldots, i_n \in \mathbb{Z}.$$

Then, in particular, $\Lambda^0 = \mathbb{Z}^n$. Just as the lattice Λ^0 acts as the group of translation symmetries of $\mathscr{C}^0$, the lattice Λ^ϵ will act on $\mathscr{C}^\epsilon$ as its symmetry group of translations. The n-cell c_v^ϵ, as well as its closure, is then defined by translating c_v^0 by the vector

$$\epsilon \left(0, i_1^v, \ldots, i_{n-1}^v\right),$$

(figure 2). It should be remarked that although there is a 1-1 correspondence between n-cells in $\mathscr{C}^0$ and $\mathscr{C}^\epsilon$ by construction, new k-cells, $k = 0, \ldots, n-1$ are created in $\mathscr{C}^\epsilon$ by subdivision of the boundary cells of each n-cell c_v^ϵ. It is not difficult to see that, indeed, $\mathscr{C}^\epsilon$ is triangle-dual for every $\epsilon \in (0, 1)$. Notice that over $\mathscr{C}^0$, each 0-cell is the common intersection of $2n$ 1-cells. The ϵ-shifts introduced to define $\mathscr{C}^\epsilon$ separate $n - 1$ of these 1-cells, leaving exactly $n + 1$ 1-cells on each old and new 0-cell. The argument with the higher dimensional cells is similar, giving the combinatorial properties that determine a triangle-dual cell decomposition. $\square$

3. Cellular path structures

Given a piecewise-smooth path $\gamma : [0, 1] \to M$, we follow the convention of denoting $\gamma(0) = s(\gamma)$ and $\gamma(1) = t(\gamma)$ (the *source* and *target* of γ).

Definition 2. We say that two piecewise-smooth paths γ and γ' in M, with $s(\gamma) = s(\gamma')$ and $t(\gamma) = t(\gamma')$, are *thinly homotopic*, if the loop $\gamma^{-1} \cdot \gamma'$ is thin, i.e. if there is a deformation retraction of $\gamma^{-1} \cdot \gamma'$ to $s(\gamma')$ for which the pullback of any 2-form in M to $[0, 1] \times [0, 1]$ is zero [3]. A homotopy of paths consisting of thinly equivalent paths is called a *retracing.*[2]

Naturally, for every thin homotopy class $[\gamma]$, there is a unique inverse class $[\gamma]^{-1}$ under path multiplication, in the sense for any representatives $\gamma \cdot \gamma^{-1}$ and $\gamma^{-1} \cdot \gamma$ are thinly null homotopic, namely, the thin homotopy class of the paths $\gamma^{-1}(t) := \gamma(1-t)$. When defined, the multiplication of thin homotopy classes of paths is associative, thus giving rise to the structure of a groupoid.

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[2]A reparametrization of a path γ is a particular kind of retracing, but the latter is considerably more general. In particular, the equivalence of piecewise-smooth paths under retracing implies that a smooth representative for any $[\gamma]$ can always be found.

FIGURE 2. The cell decomposition $\mathscr{C}^\epsilon$ in $\mathbb{R}^2$.
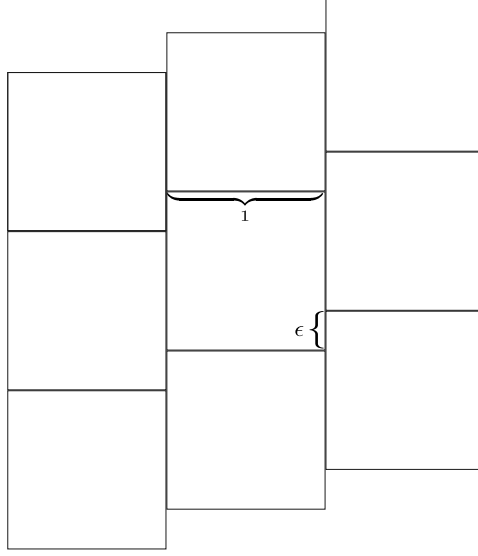
Let us fix, for every $c_\sigma \in \mathscr{C}$, a point $p_\sigma \in c_\sigma$, which will be referred as the base point of such cell. It readily follows from the previous definitions that for every k-simplex $\sigma \in N(\mathfrak{U})$, $\mathscr{U}_\sigma \setminus \{p_\sigma\}$ deformation retracts to $\partial \overline{c_\sigma}$, which is a piecewise-smooth $(n - k - 1)$-sphere. We will denote the collection of all base points in a given $\overline{c_\sigma}$ by $\mathscr{B}_\sigma$, and the collection of all base points in $\mathscr{C}$ by $\mathscr{B}_\mathscr{C}$.

Definition 3. For every $c_\sigma \in \mathscr{C}$, let $\mathscr{P}_\sigma$ be the path groupoid of piecewise-smooth paths $\gamma : [0, 1] \to \overline{c_\sigma}$, such that $s(\gamma), t(\gamma) \in \mathscr{B}_\sigma$, modulo retracing, and let $\mathscr{P}_\mathscr{C}$ be the path groupoid of piecewise-smooth paths $\gamma : [0, 1] \to M$ such that $s(\gamma), t(\gamma) \in \mathscr{B}_\mathscr{C}$, modulo retracing. In particular, each of the groupoids $\mathscr{P}_\sigma$ contains the respective loop groups $\mathscr{L}(\overline{c_\sigma}, p_{\sigma'}) = \Omega \overline{c_\sigma}(p_{\sigma'})/\sim$, consisting of classes of piecewise-smooth loops in $\overline{c_\sigma}$ based at $p_{\sigma'}$ for any $\sigma' \supset \sigma$, under retracing equivalence.[3]

Lemma 2. *Every element* $[\gamma] \in \mathscr{P}_\mathscr{C}$ *admits a minimal factorization*

$$[\gamma] = [\gamma_{v_r}] \cdot \cdots \cdot [\gamma_{v_2}] \cdot [\gamma_{v_1}],$$

with $[\gamma_{v_i}] \in \mathscr{P}_{v_i}$ *for some* $c_{v_1}, \ldots, c_{v_r} \in \mathscr{C}_n$. *Therefore, the path groupoid* $\mathscr{P}_\mathscr{C}$ *is in particular generated by the local subgroupoids* $\{\mathscr{P}_v\}_{c_v \in \mathscr{C}_n}$.

Proof. There exist multiple potential factorizations for a given path, but we prescribe a specific one, with a minimality property, as follows. Choose any representative $\gamma \in [\gamma]$ such that its image intersects the interiors of a finite

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[3]For any fixed σ and a pair $\sigma_1', \sigma_2' \supset \sigma$, the loop groups $\mathscr{L}\left(\overline{c_\sigma}, p_{\sigma_1'}\right)$ and $\mathscr{L}\left(\overline{c_\sigma}, p_{\sigma_2'}\right)$ are isomorphic under the choice of $\gamma \in \mathscr{P}_\sigma$ such that $s(\gamma) = p_{\sigma_1'}$ and $t(\gamma) = p_{\sigma_2'}$.

and *minimal* number of n-cells $c_1, \ldots, c_r \in \mathscr{C}_n$ (in the sense that there is no subcollection of n-cells with the same property, although a given n-cell may appear several times). Then, there exist subintervals $[a_1, b_1], \ldots, [a_{r-1}, b_{r-1}]$ in $[0,1]$, such that

$$\gamma([a_i, b_i]) \subset \overline{c_{v_i}} \cap \overline{c_{v_{i+1}}},$$

while for any other $t \in [0,1] \setminus \left(\cup_{i=1}^{r-1} [a_i, b_i] \right)$, $\gamma(t)$ lies in the interior of one of the previous n-cells. For every $i = 1, \ldots, r-1$, choose the minimal subcell c_{σ_i} of $\overline{c_{v_i}} \cap \overline{c_{v_{i+1}}}$ such that $\gamma([a_i, b_i]) \subset \overline{c_{\sigma_i}}$. Then, there exists another path γ', thinly equivalent to γ, with an additional explicit factorization into r subpaths

$$\gamma' = \gamma'_r \cdot \cdots \cdot \gamma'_1,$$

and for each $i = 1, \ldots, r-1$, $\gamma'_i([0,1]) \subset \overline{c_{v_i}}$, and $t(\gamma'_i) = p_{\sigma_i}$. Letting $[\gamma_{v_i}] = [\gamma'_i]$, the claim follows. $\qquad\square$

The factorization of elements in $\mathscr{P}_{\mathscr{C}}$ into local pieces motivates the introduction of local path families as essential building blocks for the study of subgroupoids in $\mathscr{P}_{\mathscr{C}}$. For any pair of cells c_σ and $c_{\sigma'}$ in $\mathscr{C}$ such that $\overline{c_\sigma} \cap \overline{c_{\sigma'}} \neq \emptyset$, an r-dimensional *local path family* $\mathscr{F}^r_{\sigma'\sigma}$, of paths γ^x satisfying $s(\gamma^x) = p_\sigma$, and $t(\gamma^x) = p_{\sigma'}$, is understood as an equivalence class of piecewise-smooth maps[4]

$$f : \overline{\mathbb{D}^r} \times [0,1] \to \overline{c_\sigma} \cup \overline{c_{\sigma'}}$$

such that $f(x,0) = p_\sigma$ and $f(x,1) = p_{\sigma'}$ for any $x \in \overline{\mathbb{D}^r}$, under simultaneous retracings, i.e., piecewise-smooth homotopy maps $H : \overline{\mathbb{D}^r} \times [0,1]^2 \to \overline{c_\sigma}$ such that for every fixed $x \in \overline{\mathbb{D}^r}$, $\gamma^x_s(t) = H(x,s,t)$ defines a thin homotopy of paths. Of special relevance will be the cases (i) when $c_\sigma, c_{\sigma'} \in \mathscr{C}_n$, and (ii) when $\overline{c_{\sigma'}} \subset \overline{c_\sigma}$.

There is a special class of local path families we are interested in, which will be called *cellular path families*, and that will be considered exclusively henceforth. Given a pair of cells such that $\overline{c_\sigma} \cap \overline{c_{\sigma'}} \neq \emptyset$ as before, we will require the path family $\mathscr{F}_{\sigma'\sigma}$ to satisfy additionally that there is a bijection between $\mathscr{F}_{\sigma'\sigma}$ and $\overline{c_\sigma} \cap \overline{c_{\sigma'}}$; in particular, it follows that $\dim \mathscr{F}_{\sigma'\sigma} = \dim \overline{c_\sigma} \cap \overline{c_{\sigma'}}$. Examples of such families can be constructed if one considers a pair of diffeomorphisms

$$\psi_\sigma : \overline{\mathbb{D}^k} \to \overline{c_\sigma} \quad \text{and} \quad \psi_{\sigma'} : \overline{\mathbb{D}^{k'}} \to \overline{c_{\sigma'}}$$

such that $\psi_{\sigma'}(0) = p_\sigma$ and $\psi_\sigma(0) = p_{\sigma'}$. The path $\gamma^x_{\sigma'\sigma}$ in the family $\mathscr{F}_{\sigma'\sigma}$, corresponding to $x \in \overline{c_\sigma} \cap \overline{c_{\sigma'}}$ is constructed as follows. Let γ^x_σ and $\gamma^x_{\sigma'}$ be the respective images in $\overline{c_\sigma}$, $\overline{c_{\sigma'}}$ under ψ_σ and $\psi_{\sigma'}$, of the linear segments in their unit ball domains, starting from 0, and such that $t(\gamma^x_\sigma) = t(\gamma^x_{\sigma'}) = x$. Then, $\gamma^x_{\sigma'\sigma} = (\gamma^x_{\sigma'})^{-1} \cdot \gamma^x_\sigma$. In the special case when $\overline{c_{\sigma'}} \subset \overline{c_\sigma}$ and $x = p_{\sigma'}$, we will simply denote $\gamma^x_{\sigma'\sigma}$ as $\gamma_{\sigma'\sigma}$.

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[4]The paths in the smooth family are defined by decreeing $\gamma^x(t) = f(x,t)$ for every fixed $x \in \overline{\mathbb{D}^r}$.

Remark 4. The cellular path families defined before satisfy the following fundamental property. For any k''-subcell $c_{\sigma''} \subset \overline{c_\sigma} \cap \overline{c_{\sigma'}}$, $k'' > 0$, consider a choice of cellular path families $\mathscr{F}_{\sigma'\sigma}$, $\mathscr{F}_{\sigma''\sigma}$, and $\mathscr{F}_{\sigma''\sigma'}$. Whenever $x \in \overline{c_{\sigma''}}$, we can induce the local factorization of thin homotopy classes

$$(3.1) \qquad\qquad [\gamma_{\sigma'\sigma}^x] = [\gamma_{\sigma''\sigma'}^x]^{-1} \cdot [\gamma_{\sigma''\sigma}^x],$$

with $[\gamma_{\sigma''\sigma}^x] \in \mathscr{F}_{\sigma''\sigma}$, $[\gamma_{\sigma''\sigma'}^x] \in \mathscr{F}_{\sigma''\sigma'}$, which could be written symbolically as

$$\mathscr{F}_{\sigma'\sigma}|_{\overline{c_{\sigma''}}} = \mathscr{F}_{\sigma''\sigma'}^{-1} \cdot \mathscr{F}_{\sigma''\sigma},$$

and which in particular applies in the case when $\overline{c_{\sigma''}} \subset \overline{c_{\sigma'}} \subset \overline{c_\sigma}$. There are several types of collections of fundamental cellular path families, connecting all base points in the elements of $\mathscr{C}$, that will be important for us. The simplest of such is given in terms of a collection

$$\mathfrak{F}_{\min} = \{\mathscr{F}_{vw}\}_{c_v, c_w \in \mathscr{C}_n, \overline{c_v} \cap \overline{c_w} \neq \emptyset},$$

where for every admissible pair $\{v, w\}$, the elements in the families $\mathscr{F}_{vw}$ and $\mathscr{F}_{wv}$ are related as

$$[\gamma_{wv}^x] = [\gamma_{vw}^x]^{-1}.$$

Any element $\mathscr{F}_{vw} \in \mathfrak{F}_{\min}$ acquires the structure of an $(n-1)$-dimensional cell in $\mathscr{P}_\mathscr{C}$, which follows from the bijective correspondence with the closed cell $\overline{c_v} \cap \overline{c_w}$, and generates an $(n-1)$-dimensional topological subgroupoid $\mathscr{P}_{\min} \subset \mathscr{P}_\mathscr{C}$, the *minimal* cellular path subgroupoid containing the families in the collection $\mathfrak{F}_{\min}$. Upon a choice of an additional factorization of the families $\mathscr{F}_{vw} = \mathscr{F}_{\tau v}^{-1} \cdot \mathscr{F}_{\tau w}$, where $\overline{c_\tau} = \overline{c_v} \cap \overline{c_w}$, the minimal path subgroupoid can be further factored into a larger $(n-1)$-dimensional topological subgroupoid

$$\mathscr{P}_{\min} \subset \mathscr{P}_{\min'} \subset \mathscr{P}_\mathscr{C},$$

namely, the minimal subgroupoid of $\mathscr{P}_\mathscr{C}$ containing the factored families $\{\mathscr{F}_{\tau v}\}_{c_v \in \mathscr{C}_n, c_\tau \in \mathscr{C}_{n-1}, \overline{c_\tau} \subset \overline{c_v}}$. In general, if we consider an arbitrary collection of cellular path families,

$$\mathfrak{F} = \{\mathscr{F}_{\sigma'\sigma}\}_{c_\sigma, c_{\sigma'} \in \mathscr{C}', \overline{c_\sigma} \cap \overline{c_{\sigma'}} \neq \emptyset}$$

indexed by an arbitrary cellular subcomplex $\mathscr{C}'$ of $\mathscr{C}$, then, there is an induced path subgroupoid $\mathscr{P}_\mathfrak{F} \subset \mathscr{P}_\mathscr{C}$ generated by all of the cellular path families in $\mathfrak{F}$. In the case when, moreover, $\mathscr{C}' = \mathscr{C}$, we will refer to the family and the path subgroupoid induced by $\mathfrak{F}$ as *complete*, and $\mathscr{P}_\mathfrak{F}$ contains a minimal path subgroupoid $\mathscr{P}_{\min} \subset \mathscr{P}_\mathfrak{F}$, the *core* of $\mathscr{P}_\mathfrak{F}$. In particular, in accordance with lemma 2, any element in $\mathscr{P}_\mathfrak{F}$ may be factored as a finite product of elements of the cellular path families $\mathscr{F}_{\sigma'\sigma}$. It is important to remark that although for different cells $c_{\sigma_1'}, c_{\sigma_2'} \subset \overline{c_\sigma}$ such that $\overline{c_{\sigma_1'}} \cap \overline{c_{\sigma_2'}} \neq \emptyset$, the families $\mathscr{F}_{\sigma_1'\sigma}$ and $\mathscr{F}_{\sigma_2'\sigma}$ are disjoint, the subcells in their boundaries corresponding to $\overline{c_{\sigma_1'}} \cap \overline{c_{\sigma_2'}}$ may be identified by means of multiplication by the "bridging" paths

$$\left[\gamma_{\sigma_1'\sigma}^x\right]^{-1} \cdot \left[\gamma_{\sigma_2'\sigma}^x\right], \qquad \left[\gamma_{\sigma_2'\sigma}^x\right]^{-1} \cdot \left[\gamma_{\sigma_1'\sigma}^x\right],$$

where $x \in \overline{c_{\sigma_1'}} \cap \overline{c_{\sigma_2'}}$. By definition, all of the cells determined by the family $\mathfrak{F}$ may be regarded as smooth cells in $\mathscr{P}_{\mathscr{C}}$.

We can summarize the construction that we have introduced in remark 4 as the following lemma.

Lemma 3. *A triangle-dual cellular decomposition $\mathscr{C}$ of M, together with a choice of a collection of cellular path families*

$$\mathfrak{F} = \{\mathscr{F}_{\sigma'\sigma}\}_{c_\sigma, c_{\sigma'} \in \mathscr{C}', \overline{c_\sigma} \cap \overline{c_{\sigma'}} \neq \emptyset}$$

parametrized by an arbitrary cell subcomplex $\mathscr{C}' \subset \mathscr{C}$, defines a topological subgroupoid $\mathscr{P}_{\mathfrak{F}}$ of the path groupoid $\mathscr{P}_{\mathscr{C}}$, generated by closed cells in correspondence with the cellular path families $\mathscr{F}_{\sigma'\sigma}$, for $c_\sigma, c_{\sigma'} \in \mathscr{C}'$ such that $\overline{c_\sigma} \cap \overline{c_{\sigma'}} \neq \emptyset$. The paths $[\gamma_{\sigma''\sigma}]$ corresponding to $c_{\sigma''} \in \mathscr{C}_0$ such that $c_{\sigma''} \subset \overline{c_\sigma}$ for $c_\sigma \in \mathscr{C}'$, determine a discrete subgroupoid $\mathscr{P}_{\mathfrak{F}}^0 \subset \mathscr{P}_{\mathfrak{F}}$.

4. CELLULAR PARALLEL TRANSPORT MAPS

Definition 4. Let G be a Lie group. A *smooth cellular parallel transport map*, relative to a choice of cellular decomposition $\mathscr{C}$ of M, is a groupoid homomorphism

$$\mathrm{PT}_{\mathscr{C}} : \mathscr{P}_{\mathscr{C}} \to G,$$

such that, for any choice of local path family $\mathscr{F}_{\sigma'\sigma}$, the induced map

$$g_{\sigma'\sigma} : \overline{c_\sigma} \cap \overline{c_{\sigma'}} \to G$$

that results from the evaluation of $\mathrm{PT}_{\mathscr{C}}$ in $\mathscr{F}_{\sigma'\sigma}$, is continuous, and its restriction to any subcell $c_{\sigma''} \subset \overline{c_\sigma} \cap \overline{c_{\sigma'}}$ is smooth. We say that two cellular parallel transport maps $\mathrm{PT}_{\mathscr{C}}$ and $\mathrm{PT}'_{\mathscr{C}}$ are equivalent if for any $c_\sigma \in \mathscr{C}$ there is an element $g_\sigma \in G$ such that

$$\mathrm{PT}'_{\mathscr{C}}([\gamma]) = g_{\sigma_2}\mathrm{PT}_{\mathscr{C}}([\gamma])g_{\sigma_1}^{-1}$$

for any $[\gamma] \in \mathscr{P}_{\mathscr{C}}$ with $s(\gamma) = p_{\sigma_1}$, and $t(\gamma) = p_{\sigma_2}$.

Remark 5. Technically speaking, there is a slightly more general way to define a smooth cellular parallel transport map, by replacing G with another grupoid in the defining homomorphism. Namely, consider a groupoid G_M, which to every point $x \in M$, assigns a G-torsor P_x, and to every morphism $x \to y$, $x, y \in M$, assigns a G-equivariant map $P_x \to P_y$, that is, a covariant functor from M, thought of as a category, to the category of G-torsors. We could then define a general parallel transport map as a groupoid homomorphism $\mathrm{PT} : \mathscr{P}_M \to G_M$, where $\mathscr{P}_M$ is the full path groupoid in M, satisfying an analogous smoothness condition. As we will see later, such homomorphism would allow to define a principal G-bundle $P \to M$. Moreover, the specialization to definition 4 will provide a collection of trivializations on the fibers over the base points of the elements in $\mathscr{C}$.

It follows as a consequence of lemma 2, that a smooth cellular parallel transport map is equivalent to a collection of groupoid homomorphisms

$$\{\mathrm{PT}_\sigma : \mathscr{P}_\sigma \to G\}_{c_\sigma \in \mathscr{C}},$$

compatible in the sense that for any $\tau \subset \sigma$ and $[\gamma] \in \mathscr{P}_\sigma \subset \mathscr{P}_\tau$, it follows that $\mathrm{PT}_\tau([\gamma]) = \mathrm{PT}_\sigma([\gamma])$, and satisfying a suitable smoothness condition. We may consider the former, or a collection of the latter, whenever it is more convenient.

Remark 6. A cellular parallel transport map $\mathrm{PT}_\mathscr{C}$ allows us to define a collection of principal G-bundles $\pi_\sigma : P_\sigma \to \overline{c_\sigma}$ for every $c_\sigma \in \mathscr{C}$. Let $\mathscr{P}_\sigma^s$ be the path space of thin homotopy classes of piecewise-smooth paths in $\overline{c_\sigma}$ whose source is $p_{\sigma'}$, for any $\sigma' \supset \sigma$. Let

(4.1) $$P_\sigma = \mathscr{P}_\sigma^s \times G/_{\sim \mathrm{PT}}$$

where two pairs $([\gamma], g)$ and $([\gamma'], g')$ are related if $t(\gamma) = t(\gamma')$ and

$$g' = \mathrm{PT}_\sigma \left([\gamma']^{-1} \cdot [\gamma] \right) g.$$

The projection of a class $[\gamma, g]$ onto M is simply defined as $[\gamma, g] \mapsto t(\gamma)$, determining a map $\pi_\sigma : P_\sigma \to \overline{c_\sigma}$. Moreover, there is a global right G-action on P_σ, defined as $[\gamma, g'] \cdot g = [\gamma, g'g]$. For every $\sigma' \supset \sigma$, there is a special point $b_{\sigma'} \in \pi_\sigma^{-1}(p_{\sigma'})$, determined by simply considering the classes $[p_{\sigma'}, e]$, where, by a slight abuse of notation, $[p_{\sigma'}]$ represents the class of the constant path at $p_{\sigma'}$. Consequently, there is a identification of the fiber $\pi_\sigma^{-1}(p_{\sigma'})$ with G.

Remark 7. A smooth and global *cellular trivialization* can be given for each principal G-bundle $\pi_\sigma : P_\sigma \to \overline{c_\sigma}$. This can be seen by considering, similarly to remark 4, a smooth family of paths $\mathscr{F}_\sigma^s \subset \mathscr{P}_\sigma^s$ with source p_σ, in bijective correspondence with $\overline{c_\sigma}$, with bijection determined by the path target map $\gamma_\sigma^x \mapsto t(\gamma_\sigma^x) = x$ (for instance, by taking any diffeomorphism $\psi_\sigma : \overline{\mathbb{D}^k} \to \overline{c_\sigma}$, for every $c_\sigma \in \mathscr{C}_k$, $1 \leq k \leq n$, such that $\psi_\sigma(0) = p_\sigma$, and the collection of paths $\{\gamma^x\}$ in $\mathscr{P}_\sigma^s$, consisting of the images of linear segments in $\overline{\mathbb{D}^k}$ with source at 0). This way, we get a bijection

$$\Psi_\sigma : \overline{c_\sigma} \times G \to P_\sigma, \qquad (x, g) \mapsto [\gamma^x, g],$$

which defines a smooth structure of manifold with boundary on P_σ, and a trivialization as a smooth principal G-bundle over $\overline{c_\sigma}$.

Since whenever $\sigma \supset \tau$, we have that $\mathscr{P}_\sigma^s \subset \mathscr{P}_\tau^s$, we can construct a smooth bijection $P_\sigma \mapsto P_\tau|_{\overline{c_\sigma}}$ by restriction, and therefore, the collection of principal G-bundles $\{P_\sigma\}_{c_\sigma \in \mathscr{C}}$ can be glued into a single bundle $\pi : P \to M$, which could also be constructed as a quotient similar to (4.1), in terms of the full path space $\mathscr{P}_\mathscr{C}^s$ of all thin homotopy equivalence classes of piecewise-smooth paths with source an arbitrary p_σ in M, and projection π defined

in a similar way. As before, such bundle would come with a preferred set of points $\{b_\sigma \in \pi^{-1}(p_\sigma)\}_{c_\sigma \in \mathscr{C}}$, which we will denote by $\mathscr{E}_{\mathscr{C}}$. Thus,

$$\pi\left(\mathscr{E}_{\mathscr{C}}\right) = \mathscr{B}_{\mathscr{C}}.$$

It is possible to give a more straightforward construction of the principal G-bundle P by means of local trivializations and transition functions. Consider an arbitrary complete collection of path families $\{\mathscr{F}_\sigma^s\}_{c_\sigma \in \mathscr{C}}$, where $\mathscr{F}_\sigma^s = \{\gamma_\sigma^x : x \in \overline{c_\sigma}\}$ as in remark 7. Then, whenever $\tau \subset \sigma$, the identity

$$[\gamma_\tau^x, g] = [\gamma_\sigma^x, g']$$

for any $x \in \overline{c_\sigma} \subset \overline{c_\tau}$, together with the corresponding trivializations, allows us to express $g' = g_{\sigma\tau}(x)g$ for some $g_{\sigma\tau}(x) \in G$. We can explicitly write

$$(4.2) \qquad g_{\sigma\tau}(x) = \mathrm{PT}_{\mathscr{C}}\left([\gamma_\sigma^x]^{-1} \cdot [\gamma_\tau^x]\right).$$

Moreover, if the collection of path families $\{\mathscr{F}_\sigma^s\}$ are changed to any other given one, the elements $g_{\sigma\tau}(x)$ would transform as

$$g_{\sigma\tau}(x) \mapsto g_\sigma(x)g_{\sigma\tau}(x)g_\tau(x)^{-1},$$

for some well-defined smooth functions $g_\sigma : \overline{c_\sigma} \to G$ and $(g_\tau : \overline{c_\tau} \to G)|_{\overline{c_\sigma}}$.

Definition 5. The *glueing maps* induced from a cellular parallel transport map $\mathrm{PT}_{\mathscr{C}}$ and a complete collection of cellular path families $\mathfrak{F}$, for a flag $\overline{c_\sigma} \subset \overline{c_\tau}$ in $\mathscr{C}$,[5] are the $g_{\sigma\tau} : \overline{c_\sigma} \to G$ defined above. Clearly, the glueing maps satisfy the factorization property

$$(4.3) \qquad \left(g_{\sigma\tau}|_{\overline{c_{\sigma'}}}\right) = g_{\sigma'\sigma}^{-1} \cdot g_{\sigma'\tau}$$

whenever $\tau \subset \sigma \subset \sigma'$. The *clutching maps*, or *skeletal transition functions*, for a minimal collection $\mathfrak{F}_{\min}$ (remark 4), are the maps of the form

$$(4.4) \qquad h_{vw}(x) = \mathrm{PT}_{\mathscr{C}}\left([\gamma_{vw}^x]\right) = g_{\tau v}^{-1} \cdot g_{\tau w}, \qquad [\gamma_{vw}^x] \in \mathscr{F}_{vw}$$

for any $x \in \overline{c_v} \cap \overline{c_w}$, where $v, w \subset \tau$, for some $c_\tau \in \mathscr{C}_{n-1}$.

Clearly, the clutching maps satisfy the relation $h_{wv} = h_{vw}^{-1}$. Moreover, when cyclically oriented triples are taken into account (remarks 2 and 3), whenever $\overline{c_{v_1}} \cap \overline{c_{v_2}} \cap \overline{c_{v_3}} \neq \emptyset$, and (2.2) is satisfied for $c_\sigma \in \mathscr{C}_{n-2}$, these maps satisfy the cocycle condition

$$h_{v_1 v_2} h_{v_2 v_3} h_{v_3 v_1}|_{\overline{c_\sigma}} = e.$$

As we will see in section 5, the maps $\{h_{vw} : \overline{c_\tau} \to G\}_{v,w \subset \tau,\, c_\tau \in \mathscr{C}_{n-1}}$ play the role of transition functions for the principal bundle P. Indeed, it is easy to verify that they transform as transition functions under equivalence of cellular parallel transport maps. The latter is stated in precise terms in the next proposition.

---

[5] When complete cellular path families are considered, it is enough to define the glueing maps over flags of length 2 in $\mathscr{C}$, as any other glueing map could be induced from those.

Remark 8. A *horizontal lift to P*, for every path $\gamma : [0,1] \to M$ and a choice of initial condition $[\gamma', g] \in \pi^{-1}(\gamma(0))$ can be canonically constructed from the cellular parallel transport map $\mathrm{PT}_\mathscr{C}$. Namely, horizontal lifts can first be defined if we take any path $\gamma : [0,1] \to M$ and an initial condition $[\gamma', g]$, where $[\gamma'] \in \mathscr{P}^s_\mathscr{C}$. Then by the independence under reparametrization, a path $\lambda(\gamma) : [0,1] \to P_\sigma$ can be defined as

$$[\lambda(\gamma)](\epsilon) = \left[(\gamma \cdot \gamma') \,|_{[0,(\epsilon+1)/2]}, g \right],$$

for $\epsilon \in [0,1]$, with source $s(\lambda(\gamma)) = [\gamma', g]$, satisfying $\pi \circ \lambda(\gamma)(\epsilon) = \gamma(\epsilon)$. Horizontal lifts may also be constructed as an iteration of local steps, if we recall the factorization of a class of piecewise smooth paths in M in lemma 2. The local cellular horizontal lifts on each $\overline{c_{\sigma_i}}$, $i = 1, \dots, r$, together with an iteration of subsequent initial conditions, determines the full horizontal lift of γ to P.

Proposition 1. *Let us fix a triangle-dual cellular decomposition $\mathscr{C}$ in M.*

(i) *If two different cellular parallel transport maps $\mathrm{PT}_\mathscr{C}$ and $\mathrm{PT}'_\mathscr{C}$ are equivalent, then there is a set of constants $\{g_\sigma\}_{c_\sigma \in \mathscr{C}}$, in such a way that the glueing maps $\{g_{\sigma'\sigma}\}_{\overline{c_\sigma} \cap \overline{c_{\sigma'}} \neq \emptyset}$ and $\{g'_{\sigma'\sigma}\}_{\overline{c_\sigma} \cap \overline{c_{\sigma'}} \neq \emptyset}$, induced from a fixed choice of complete cellular path families $\mathfrak{F} = \{\mathscr{F}_{\sigma'\sigma}\}$, are related as*

$$g'_{\sigma'\sigma}(x) = g_{\sigma'} \cdot g_{\sigma'\sigma}(x) \cdot g_\sigma^{-1}.$$

(ii) *If the choice of $\mathfrak{F}$ is changed, then there is a set of smooth maps $\{g_\sigma(x) : \overline{c_\sigma} \to G\}_{c_\sigma \in \mathscr{C}}$, in such a way that the maps $\{g_{\sigma'\sigma}\}_{\overline{c_\sigma} \cap \overline{c_{\sigma'}} \neq \emptyset}$ induced from $\mathrm{PT}_\mathscr{C}$ transform pointwise as*

$$g_{\sigma'\sigma}(x) \mapsto g_{\sigma'}(x) g_{\sigma'\sigma}(x) g_\sigma(x)^{-1}.$$

Proof. (i) is a straightforward consequence of definition 4. Namely, the value of $g_{\sigma'\sigma}(x)$ (and similarly for the value $g'_{\sigma'\sigma}(x)$), for an arbitrary $x \in \overline{c_\sigma} \cap \overline{c_{\sigma'}}$, is equal to $\mathrm{PT}_\mathscr{C}\left(\left[\gamma^x_{\sigma'\sigma} \right] \right)$. (ii) has been discussed already. It should be remarked that the proposition applies in particular to the clutching maps $h_{vw} = g_{vw}$. $\qquad\square$

In fact, it will turn out that only the homotopy type of the functions $\{g_{\sigma'\sigma}\}$, in a suitable sense, will be relevant to determine an equivalence class of principal G-bundles on M. Such notion of homotopical *cellular bundle data*, characterizing a principal bundle with trivializations over $\mathscr{C}$, up to equivalence, is axiomatized in definition 10.

Remark 9. It is also possible to construct an honest system of transition functions for the principal bundle $\pi : P \to M$ if each of the local bundles P_σ is constructed instead over the corresponding open set $\mathscr{U}_\sigma$, by extending the spaces $\mathscr{P}^s_\sigma$ to consist of paths belonging to $\mathscr{U}_\sigma$ (and not only to $\overline{c_\sigma}$), and repeating the previous constructions verbatim.

Finally, we remark that the notion of a cellular parallel transport map, which we have axiomatized before, together with the horizontal lift property described in remark 8, implies the existence of an equivalence class of smooth connections in the principal G-bundle P, under the relation determined by the action of the group of bundle automorphisms covering the identity map in M, and acting as the identity over the fibers $\{\pi^{-1}(p_\sigma)\}_{c_\sigma \in \mathscr{C}}$, which we will refer to as *restricted gauge transformations*.

We summarize our findings in the form of a theorem. We have provided a rigorous proof of theorem 2 in [21], which is a cellular analog to Barrett's reconstruction theorem [3]. Reformulated in our terminology, the statement is the following.

Theorem 1 (Reconstruction theorem)**.** *Let M be an oriented n-manifold, equipped with a triangle-dual cellular decomposition $\mathscr{C}$, and G a Lie group. There is a bijective correspondence*

$$\left\{ \begin{array}{c} G\text{-valued cellular} \\ parallel\ transport \\ maps\ \mathrm{PT}_\mathscr{C}\ in\ M,\ up \\ to\ equivalence \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} smooth\ principal\ G\text{-bundles}\ P \to M \\ with\ a\ choice\ of\ fiber\ points\ \mathscr{E}_\mathscr{C},\ up\ to \\ equivalence,\ and\ a\ smooth\ connection \\ up\ to\ restricted\ gauge\ equivalence \end{array} \right\}$$

and moreover, for any cellular parallel transport map $\mathrm{PT}_\mathscr{C}$ with induced principal G-bundle P, there is an injective correspondence

$$\left\{ \begin{array}{c} Path\ families\ \mathscr{F}_\sigma^s, \\ for\ c_\sigma \in \mathscr{C} \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} Local\ trivializations\ of \\ P\ over\ \overline{c_\sigma},\ for\ c_\sigma \in \mathscr{C} \end{array} \right\}$$

We will finish the present section with definition 8, the main concept on which the rest of the article will be built on. Such definition is cemented in the following couple of definitions regarding discrete topological structures in the path groupoid $\mathscr{P}_\mathscr{C}$, and provides a justification for the insufficiency of the standard lattice gauge theory (LGT) data to capture the global topology of a principal G-bundle.

Definition 6. A *cellular network* Γ in a triangle-dual cellular decomposition $\mathscr{C}$ of M, relative to a collection of cell base points $\mathscr{B}_\mathscr{C} = \{p_\tau \in c_\tau\}_{c_\tau \in \mathscr{C}}$, is a collection of nonintersecting paths $\gamma_{\sigma\tau} \subset \overline{c_\tau}$ for every $c_\tau \in \mathscr{C}$, and every 0-cell $c_\sigma = p_\sigma \in \partial \overline{c_\tau}$, joining p_τ and p_σ. We will denote by $\mathscr{P}_\Gamma^0$ the discrete groupoid of the path groupoid $\mathscr{P}_\mathscr{C}$ generated by a cellular network Γ, or by $\mathscr{P}_\mathfrak{F}^0$ if Γ is induced by a complete collection of cellular path families $\mathfrak{F}$.

Definition 7. A *standard lattice gauge field*, relative to a cellular parallel transport map $\mathrm{PT}_\mathscr{C}$ and a cellular network $\Gamma = \{\gamma_{\sigma\tau}\}_{p_\sigma \in \overline{c_\tau}, c_\tau \in \mathscr{C}}$, is an assignment of a group element $\mathrm{PT}_\mathscr{C}([\gamma_{\sigma\tau}])$ for every path $\gamma_{\sigma\tau}$, or equivalently, a groupoid homomorphism $\mathrm{PT}_\Gamma^0 : \mathscr{P}_\Gamma^0 \to G$.

Definition 8. Let M be an oriented n-dimensional manifold, $n \geq 2$, with a triangle-dual cellular decomposition $\mathscr{C}$, a complete collection $\mathfrak{F}$ of cellular path families (remark 4), and its induced path subgroupoid $\mathscr{P}_\mathfrak{F}$. By an *extended lattice gauge field*, we mean an equivalence class of cellular parallel

transport maps $\{\mathrm{PT}_\mathscr{C}\}_\mathfrak{F}$, where $\mathrm{PT}_\mathscr{C} \sim \mathrm{PT}'_\mathscr{C}$ if their restrictions to the path subgroupoid $\mathscr{P}_\mathfrak{F} \subset \mathscr{P}_\mathscr{C}$ are homotopic, relative to a fixed choice of standard lattice gauge field over its discrete subgroupoid $\mathscr{P}^0_\mathfrak{F}$.

Remark 10. In fact, the definition of an extended lattice gauge field does not require to completely fix a specific collection of cellular path families $\mathfrak{F}$, since any other collection $\mathfrak{F}'$ whose induced discrete subgroupoid $\mathscr{P}^0_{\mathfrak{F}'}$ coincides with $\mathscr{P}^0_\mathfrak{F}$ would determine the same homotopy class of cellular parallel transport maps, i.e., $\{\mathrm{PT}_\mathscr{C}\}_\mathfrak{F} = \{\mathrm{PT}_\mathscr{C}\}_{\mathfrak{F}'}$. In such case, the path subgroupoids $\mathscr{P}_\mathfrak{F}$ and $\mathscr{P}_{\mathfrak{F}'}$ are smoothly homotopic, and by definition, there is a 1-1 correspondence between smooth homotopies of subgroupoids $\mathscr{P}_\mathfrak{F}$ fixing $\mathscr{P}^0_\mathfrak{F}$ and smooth homotopies of collections of cellular path framilies $\mathfrak{F}$ with common, fixed paths connecting base points to 0-boundary cells. Since there is only one such homotopy class for every pair of intersecting cells $c_\sigma, c_{\sigma'} \in \mathscr{C}$, we could think of the homotopy class of path subgroupoids $\{\mathscr{P}_\mathfrak{F}\}$, relative to a fixed choice of discrete subgroupoid $\mathscr{P}^0_\Gamma$, as an intrinsic object associated to $\mathscr{C}$ and $\mathscr{P}^0_\Gamma$. Hence, an extended gauge field is also an intrinsic object, depending only of $\mathscr{C}$ and a choice of standard lattice gauge field PT^0_Γ.

5. Extended lattice gauge fields and principal bundles

Let us now assume for simplicity that G is a *connected* Lie group. The extra complications of the general case are easy to sort out, as the connected component of the identity G_0 is normal in G, and $\pi_0(G, e) \cong G/G_0$.

We have seen that a special characteristic feature that distinguishes a triangle-dual cellular decomposition $\mathscr{C}$ on an oriented n-manifold M, is that every triple $\{c_{v_1}, c_{v_2}, c_{v_3}\} \subset \mathscr{C}_n$ intersecting nontrivially, determines a triple $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\} \subset \mathscr{C}_{n-1}$ and a unique $c_\sigma \in \mathscr{C}_{n-2}$, together with a collection of gapless flags of oriented cells

$$
\text{(5.1)} \qquad
\begin{array}{ccccc}
 & & \overline{c_{v_1}} & & \\
 & \subset & & \supset & \\
\overline{c_{\tau_3}} & \supset & \overline{c_\sigma} & \subset & \overline{c_{\tau_2}} \\
 & \subset & \cap & & \subset \\
\overline{c_{v_2}} & \supset & \overline{c_{\tau_1}} & \subset & \overline{c_{v_3}}
\end{array}
$$

a structure that is invariant under index permutations (remarks 2 and 3), and conversely, such c_σ, together with an additional choice of orientation, determines the triples $\{c_{v_1}, c_{v_2}, c_{v_3}\}$ and $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\}$, up to permutations of the indices. An orientation of c_σ corresponds to a choice of gapless flag in the collection, up to the action of the alternating group A_3 of cyclic permutations. An arbitrary collection of clutching maps, relative to $\mathscr{C}$,

$$
\{h_{vw}\}_{\overline{c_v} \cap \overline{c_w} \neq \emptyset, \, c_v, c_w \in \mathscr{C}_n}
$$

must satisfy a similar invariance property under a suitably defined S_3-action, for every triple of n-cells as above, when the corresponding triple of maps

are restricted to their common $(n-2)$-cell closure $\overline{c_\sigma}$. Such S_3-action can be described as follows. There is an action of S_3 on $G \times G \times G$, prescribed on a choice of two generators in the following way: for the (even) 3-cycle (123), $(123) \cdot (g_1, g_2, g_3) = (g_3, g_1, g_2)$, and for the (odd) transposition (12), $(12) \cdot (g_1, g_2, g_3) = (g_2^{-1}, g_1^{-1}, g_3^{-1})$. Now, consider the multiplication map

$$T_G : G \times G \times G \to G, \qquad (g_1, g_2, g_3) \mapsto g_1 g_2 g_3,$$

let $V_G = T_G^{-1}(e)$, and denote by pr_i the projections from $G \times G \times G$ into the ith factor. In particular, it readily follows that V_G is invariant under the S_3-action defined before, and that $V_G \cong G \times G$ under the projections $\mathrm{pr}_i \times \mathrm{pr}_j$ to any pair of distinct $i, j \in \{1, 2, 3\}$. The previous action is faithful, and we will refer to it as the *triadic* action for the group G.

Consider now any triple of n-cells $\{c_{v_1}, c_{v_2}, c_{v_3}\}$ as before, together with its collection of clutching maps, which is parametrized by the ordered triples (ijk), $i, j, k \in \{1, 2, 3\}$, $i \neq j$, $k \neq i, j$. Define the maps

$$\mathbf{h}_\sigma^{ijk} := \left(h_{v_i v_j}|_{\overline{c_\sigma}} \right) \times \left(h_{v_j v_k}|_{\overline{c_\sigma}} \right) \times \left(h_{v_k v_i}|_{\overline{c_\sigma}} \right) : \overline{c_\sigma} \times \overline{c_\sigma} \times \overline{c_\sigma} \to G \times G \times G$$

which satisfy that their restriction to the diagonal $\Delta(\overline{c_\sigma} \times \overline{c_\sigma} \times \overline{c_\sigma})$ lies in V_G. The fact that the labeling in the triple of n-cells is actually arbitrary is equivalent to the fact that the S_3-action on the set of maps

$$\left\{ \mathbf{h}_\sigma^{ijk} \right\}$$

given by index permutations, is determined by postcomposition with the triadic action on their respective images. Thus, we can, and we will, fix an arbitrary labeling on each intersecting triple of n-cells.

Definition 9. A continuous map $h : \overline{c_\tau} \to G$ is said to be *cellularly-smooth* if its restriction to any subcell $c_\sigma \subset \overline{c_\tau}$ is smooth. For any triple $\{c_{v_1}, c_{v_2}, c_{v_3}\} \subset \mathscr{C}_n$ as in (5.1), we say that two triples of cellularly-smooth maps

$$(h_{v_1 v_2}, h_{v_2 v_3}, h_{v_3 v_1}), (h'_{v_1 v_2}, h'_{v_2 v_3}, h'_{v_3 v_1}) : \overline{c_{\tau_3}} \times \overline{c_{\tau_1}} \times \overline{c_{\tau_2}} \to G \times G \times G$$

whose restriction to the diagonal $\Delta(\overline{c_\sigma} \times \overline{c_\sigma} \times \overline{c_\sigma})$ lies in V_G, are *cellularly equivalent* if there is a homotopy of cellularly-smooth maps

$$(h_{v_1 v_2}(t),\ h_{v_2 v_3}(t),\ h_{v_3 v_1}(t)) : \overline{c_{\tau_3}} \times \overline{c_{\tau_1}} \times \overline{c_{\tau_2}} \times [0, 1] \to G \times G \times G$$

between them, whose restriction to $\Delta(\overline{c_\sigma} \times \overline{c_\sigma} \times \overline{c_\sigma})$ lies in V_G for all $t \in (0, 1)$.

The following definition is the cornerstone that allows us to recast the notion of an equivalence class of principal G-bundles in terms of cellular homotopies.

Definition 10. Let M be an oriented n-dimensional manifold, $n \geq 2$, and $\mathscr{C}$ a triangle-dual cellular decomposition of it. By a choice of *homotopical cellular bundle data* $\mathscr{D}$, relative to $\mathscr{C}$, we mean an equivalence class of collections of cellularly smooth maps

$$\left[\{ h_{vw} : \overline{c_\tau} \to G \}_{v, w \subset \tau,\ c_\tau \in \mathscr{C}_{n-1}} \right],$$

such that for each for each triple $\{c_{v_1}, c_{v_2}, c_{v_3}\} \subset \mathscr{C}_n$ intersecting nontrivially, the restriction $h_{v_1 v_2} \times h_{v_2 v_3} \times h_{v_3 v_1}|_{\Delta(\overline{c_\sigma} \times \overline{c_\sigma} \times \overline{c_\sigma})}$ lies in V_G. Two collections are equivalent if for any triple $\{c_{v_1}, c_{v_2}, c_{v_3}\} \subset \mathscr{C}_n$ intersecting nontrivially, the corresponding triples of maps are cellularly equivalent.

Remark 11. We have excluded the case $n = 1$ in definition 10, since the notion of bundle data is vacuous in such case. Such omission is irrelevant, since principal bundles are always trivial over any 1-dimensional smooth manifold, at least when G is connected.

The homotopical data of definition 10 is equivalent to an equivalence class of principal G-bundles: a proof of this fact is provided in theorem 5 in appendix B. The relevance of this characterization of an equivalence class of principal G-bundles comes from the fact that it is suited to the study of the cellular parallel transport maps of section 4, leading us to define a finer set of homotopical data, equivalent to the extended lattice gauge theory (ELGT) data of definition 8, giving rise to sufficient conditions to capture the homotopy type of all cellular parallel transport maps yielding isomorphic principal G-bundles. Such data is presented in theorem 2, and can be described as the result of dissecting a homotopy class of collections of clutching maps, relative to a fixed set of values over all 0-cells in $\mathscr{C}$, and which also determine a well-defined collection of homotopical cellular bundle data. The missing step to establish such characterization is the glueing mechanism, which we now describe in lemma 4. For a finite subset $X \subset S^k$ and a map $g_0 : X \to G$, let $[S^k, G, g_0]$ denote the space of homotopy classes of piecewise smooth maps $g : S^k \to G$ such that $g|_X = g_0$.

Lemma 4. *Let $\mathscr{C}$ be a triangle-dual cellular decomposition of the k-sphere S^k. If any $c_\tau \in \mathscr{C}_k$ has assigned a homotopy class of cellularly smooth maps $[g_\tau : \overline{c_\tau} \to G]$ whose values at any 0-cell $c_{\sigma''}$ are fixed, and such that for any $c_{\tau_1}, c_{\tau_2} \in \mathscr{C}_k$ with $\overline{c_\sigma} = \overline{c_{\tau_1}} \cap \overline{c_{\tau_2}} \neq \emptyset$, we have that $[g_{\tau_1}|_{\overline{c_\sigma}}] = [g_{\tau_2}|_{\overline{c_\sigma}}]$ as homotopy classes relative to the 0-cells in $\overline{c_\sigma}$, then the classes $\{[g_\tau]\}_{c_\tau \in \mathscr{C}_k}$ can be glued into a well-defined element $[g] \in [S^k, G, g_0]$, with $g_0 : \mathscr{C}_0 \to G$ the induced fixed map on 0-cells.*

Proof. Upon introducing the glueing mechanism, which is the fundamental step to integrate neighboring classes of cellularly equivalent maps to a new class over the union of their domains, and then repeating it as many times as necessary, it is possible to obtain an honest homotopy class in $\pi_k(G, g_\tau(c_{\sigma''}))$. Such procedure will finish after a finite number of steps.

Consider an arbitrary k-cell, which we denote c_{τ_1}, together with all k-cells neighboring c_{τ_1}, that is, the $c_{\tau'} \in \mathscr{C}_k$ such that $\overline{c_{\tau_1}} \cap \overline{c_{\tau'}} \neq \emptyset$, which can also be labeled as $c_{\tau_2}, \ldots, c_{\tau_{m_1}}$. The first step of the glueing procedure is to construct a class of cellularly equivalent maps defined over the closed set $\cup_{i=1}^{m_1} \overline{c_{\tau_i}}$. Such set is either homeomorphic to the closed disk $\overline{\mathbb{D}^k}$, to S^k minus a finite number of open disks, or to S^k, and has an induced orientation. We will consider each case, as the first one eventually leads to the last two.

Let us first assume that $\cup_{i=1}^{m_1}\overline{c_{\tau_i}}$ is a closed disk in S^k. By hypothesis, for any pair $\{c_{\tau_1}, c_{\tau_i}\}$ with $\overline{c_{\tau_1}} \cap \overline{c_{\tau_i}} = \overline{c_{\sigma_{1i}}}$, the induced classes $[g_{\tau_1}|_{\overline{c_{\sigma_{1i}}}}]$ and $[g_{\tau_i}|_{\overline{c_{\sigma_{1i}}}}]$ coincide. Since there are no local obstructions over multiple intersections, we may choose representatives such that $g_{\tau_1}|_{c_{\tau_{1i}}} = g_{\tau_i}|_{c_{\tau_{1i}}}$, and moreover, we can apply the same principle to all intersections $\overline{c_{\tau_{ij}}} = \overline{c_{\tau_i}} \cap \overline{c_{\tau_j}}$, $2 \leq i, j \leq m_1$, we may also assume that $g_{\tau_i}|_{c_{\tau_{ij}}} = g_{\tau_j}|_{c_{\tau_{ij}}}$. Hence, altogether, the representatives $g_{\tau_1}, \ldots, g_{\tau_{m_1}}$ glue to define a piecewise-smooth function on $\cup_{i=1}^{m_1}\overline{c_{\tau_i}}$, and hence induce a class of cellularly equivalent maps on $\cup_{i=1}^{m_1}\overline{c_{\tau_i}}$. We now add all k-cells $c_{\tau''}$ intersecting nontrivially with $\cup_{i=1}^{m_1}\overline{c_{\tau_i}}$, and repeat the previous procedure, until the complementary k-cells do not intersect pairwise. Then, the resulting set must be S^k minus a finite number of open disks. Call the remaining cells $c_{\tau_{f_1}}, \ldots, c_{\tau_{f_l}}$. Upon ordering the closures of $(k-1)$-cells c_σ in the boundary of each $\overline{c_{\tau_{f_j}}}$, there is a representative $g_{\tau_{f_j}}$ in its class whose values at each $\overline{c_\sigma}$ coincide with the boundary values of the previously constructed g. This way, we obtain a piecewise-smooth map $g : S^k \to G$, up to piecewise-smooth homotopy fixing the values at every 0-cell, whose restriction to any k-cell recovers the starting homotopy classes.

It remains to check that the previous procedure is independent of the choice of k-cell in S^k at every step. Repeat the procedure with any other choice of k-cells at every step, and call g' any map constructed in such a way. Since g' attains the same values than g at any 0-cell, and by hypothesis, the restriction of $g'g^{-1} : S^k \to G$ to any of the k-cells of $\mathscr{C}$ would determine a trivial cellular homotopy class, it follows that the homotopy class of $g'g^{-1}$, as an element in $\pi_k(G, e)$, must be trivial. Then, in particular, it follows that $[g'] = [g]$ as elements in $[S^k, G, g_0]$. $\square$

Theorem 2 (Dissection of extended lattice gauge fields)**.** *An extended lattice gauge field* $\{\mathrm{PT}_\mathscr{C}\}$ *on* $(M, \mathscr{C})$ *is equivalent to a standard lattice gauge field* $\mathrm{PT}_\Gamma^0 : \mathscr{P}_\Gamma^0 \to G$, *together with a homotopy class of collections of glueing maps* $\{\{g_{\sigma\tau}\}_{\overline{c_\sigma} \subset \overline{c_\tau}}\}$ *defined in* (4.2), *with fixed values at 0-cells prescribed by* PT_Γ^0. *More explicitly, an extended lattice gauge field is equivalent to* PT_Γ^0 *and a map which assigns, to every flag* $\overline{c_\sigma} \subset \overline{c_\tau}$ *in* $\mathscr{C}$, $c_\sigma \notin \mathscr{C}_0$, *the following collection of local homotopy data of glueing maps:*
(a) *To every 0-subcell* $\overline{c_{\sigma''}} \subseteq \overline{c_\sigma}$, *a group element*

$$g_{\sigma\tau}(p_{\sigma''}) = \mathrm{PT}_\Gamma^0 \left([\gamma_{\sigma''\sigma}]^{-1} \cdot [\gamma_{\sigma''\tau}] \right) \in G,$$

(b) *More generally, to every k-subcell* $\overline{c_{\sigma''}} \subseteq \overline{c_\sigma}$, $k > 0$, *an extension class of maps* $[g_{\sigma\tau}|_{\overline{c_{\sigma''}}}]$ *from* $\partial\overline{c_{\sigma''}}$ *to* $\overline{c_{\sigma''}}$, *given the inductive boundary homotopy constraint in* $[S^{k-1}, G, g_{\partial\overline{c_{\sigma''}}}]$, *when $k \geq 2$,*[6]

$$\left[g_{\sigma\tau}|_{\partial\overline{c_{\sigma''}}}\right] = \left[\sum_{\{\sigma''' \in \mathscr{C}_{k-1} \,|\, \sigma''' \supset \sigma''\}} g_{\sigma\tau}|_{\overline{c_{\sigma'''}}}\right],$$

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[6]The sum in the second term of the equality denotes the glueing of homotopy classes of cellularly smooth maps to S^{k-1} from lemma 4, for the map $g_{\partial\overline{c_{\sigma''}}}$ of values at 0-cells.

which is regarded to be trivial in the free homotopy class it spans. Moreover, the assignment is such that whenever $\sigma' \supset \sigma \supset \tau$, the compatibility condition

$$g_{\sigma'\tau}(p_{\sigma''}) = g_{\sigma'\sigma}(p_{\sigma''}) \cdot g_{\sigma\tau}(p_{\sigma''}),$$

is satisfied at every 0-cell $c_{\sigma''} \subset \overline{c_{\sigma'}}$, and more generally, the compatibility condition

$$\left[g_{\sigma'\tau}|_{\overline{c_{\sigma''}}} \right] = \left[\left(g_{\sigma'\sigma}|_{\overline{c_{\sigma''}}} \right) \cdot \left(g_{\sigma\tau}|_{\overline{c_{\sigma''}}} \right) \right], {}^{7}$$

is satisfied at every k-cell $c_{\sigma''} \subseteq \overline{c_{\sigma'}}$, $k = 1, \ldots, \dim(c_{\sigma'})$.

Proof. Consider an arbitrary extended lattice gauge field $\{\mathrm{PT}_{\mathscr{C}}\}$. Each class representative $\mathrm{PT}_{\mathscr{C}}$ induces a collection of glueing maps $\{g_{\sigma\tau}\}$, as the result of evaluating $\mathrm{PT}_{\mathscr{C}}$ at the family $\mathscr{F}_{\sigma\tau}$. All the values $g_{\sigma\tau}(p_{\sigma''})$ at any 0-cell $c_{\sigma''}$ for a given $\overline{c_{\sigma}} \subset \overline{c_{\tau}}$ coincide, as they correspond to the evaluations of PT_{Γ}^{0} at the elements $\left[\gamma_{\sigma''\sigma}^{-1} \right] \cdot [\gamma_{\sigma''\tau}]$ in the discrete groupoid $\mathscr{P}_{\Gamma}^{0}$. Hence, the homotopy class $\{\mathrm{PT}_{\mathscr{C}}\}$ is equivalent to a homotopy class of collections of glueing maps $\{\{g_{\sigma\tau}\}_{\overline{c_{\sigma}} \subset \overline{c_{\tau}}}\}$ with fixed values over $\mathscr{C}_{0}$. The disection of $\{\mathrm{PT}_{\mathscr{C}}\}$ as relative extension homotopy classes with fixed values over 0-cells is then straightforward. The restriction of the class representatives $g_{\sigma\tau}$ to any subcell $\overline{c_{\sigma''}}$ determines local homotopy classes of maps over $\overline{c_{\sigma''}}$ relative to the 0-cells in its boundary, that can be factored as a necessarily trivial boundary homotopy class over $\partial\overline{c_{\sigma''}}$, together with an extension class to the interior $c_{\sigma''}$. Clearly, the classes associated to $(k-1)$-cells $\overline{c_{\sigma'''}}$ in $\partial\overline{c_{\sigma''}}$, for any given k-cell $c_{\sigma''}$, glue according to lemma 4 to a trivial class over $\partial\overline{c_{\sigma''}}$. The homotopical data obtained this way clearly satisfies the compatibility conditions stated above.

The converse is verified in a similar way: as a consequence of lemma 4, given a collection of homotopy classes of extension maps to $c_{\sigma''}$ for every flag $\overline{c_{\sigma''}} \subset \overline{c_{\sigma}} \subset \overline{c_{\tau}}$, with fixed values at 0-cells, and satisfying the glueing compatibility conditions above, we can reconstruct a homotopy class of collections of glueing maps $\{\{g_{\sigma\tau}\}_{\overline{c_{\sigma}} \subset \overline{c_{\tau}}}\}$, relative to certain fixed values over their 0-cells, which is in turn equivalent to an extended lattice gauge field $\{\mathrm{PT}_{\mathscr{C}}\}$. $\qquad\square$

Recall that every path subgroupoid $\mathscr{P}_{\mathfrak{F}}$ has a special minimal subgroupoid $\mathscr{P}_{\min}$ generated by the cellular path subfamilies $\mathfrak{F}_{\min}$, which consist of those paths generated by element in $\mathfrak{F}$, whose source and target are base points of neighboring n-cells (remark 4). We can define larger homotopy classes of cellular parallel transport maps $\{\mathrm{PT}_{\mathscr{C}}\}_{\min}$, with fixed values over the discrete subgroupoid $\mathscr{P}_{\min}^{0} = \mathscr{P}_{\mathfrak{F}}^{0} \cap \mathscr{P}_{\min}$, and consequently, a projection $\pi_{\min}(\{\mathrm{PT}_{\mathscr{C}}\})$ for every extended lattice gauge field onto the larger class that contains it.

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[7]The right-hand side denotes the class of products of piecewise-smooth representatives in the relative extension classes $\left[\left(g_{\sigma'\sigma}|_{\overline{c_{\sigma''}}} \right) \right]$, $\left[\left(g_{\sigma\tau}|_{\overline{c_{\sigma''}}} \right) \right]$, which clearly coincides with the corresponding relative extension class of any class representative.

Definition 11. The *core* of an extended lattice gauge field $\{\mathrm{PT}_{\mathscr{C}}\}$ is the homotopy class of cellular parallel transport maps with respect to $\{\mathfrak{F}_{\min}\}$ containing $\{\mathrm{PT}_{\mathscr{C}}\}$, that is, $\{\mathrm{PT}_{\mathscr{C}}\}_{\min} := \pi_{\min}(\{\mathrm{PT}_{\mathscr{C}}\})$.

Corollary 1. *The core $\{\mathrm{PT}_{\mathscr{C}}\}_{\min}$ of an extended lattice gauge field $\{\mathrm{PT}_{\mathscr{C}}\}$ is equivalent to a collection of secondary local homotopy data, of the following form:*

(a) *To every triple of elements $c_{v_1}, c_{v_2}, c_{v_3} \in \mathscr{C}_n$ such that $\overline{c_{v_1}} \cap \overline{c_{v_2}} \cap \overline{c_{v_3}} = \overline{c_\sigma}$ with $c_\sigma \in \mathscr{C}_{n-2}$ as in* (5.1), *we assign*

  (i) *For every 0-cell $c_{\sigma_0''} \subset \overline{c_\sigma}$, a point $\mathbf{h}_\sigma(c_{\sigma_0''}) \in V_G$,*

  (ii) *More generally, for every k-cell $c_{\sigma'} \subseteq \overline{c_\sigma}$, $k = 1, \ldots, n-2$, a relative homotopy class of maps*

$$\left[\mathbf{h}_\sigma|_{\overline{c_{\sigma'}}} : \overline{c_{\sigma'}} \to V_G \right],$$

  with fixed values over 0-subcells, representing an extension class from $\partial \overline{c_{\sigma'}}$ to $\overline{c_{\sigma'}}$, that is determined by the inductive boundary data in $\left[S^{k-1}, V_G, \mathbf{h}_{\partial \overline{c_{\sigma'}}} \right]$ when $k \geq 2$,

$$\left[\mathbf{h}_\sigma|_{\partial \overline{c_{\sigma'}}} \right] = \left[\sum_{\{c_{\sigma''} \in \mathscr{C}_{k-1} \,:\, \sigma'' \supset \sigma'\}} \mathbf{h}_\sigma|_{\overline{c_{\sigma''}}} \right], {}^{8}$$

  whose induced free homotopy class is trivial.

The assignment is equivariant for the permutation action of the group S_3 on the triple $c_{v_1}, c_{v_2}, c_{v_3}$ and its triadic action on V_G.

(b) *To every pair of elements $c_v, c_w \in \mathscr{C}_n$ such that $\overline{c_v} \cap \overline{c_w} = \overline{c_\tau}$ with $c_\tau \in \mathscr{C}_{n-1}$, we assign an extension class $[h_{vw} : \overline{c_\tau} \to G]$ of the inductively glued boundary class over $\partial \overline{c_\tau}$ to $\overline{c_\tau}$, in such a way that the resulting induced glued class $[h_{wv} \cdot h_{vw}]$ is homotopically trivial.*

Proof. Readily follows from theorem 2, once we observe that there is an the equivalence between the core of a lattice gauge field and a homotopy class of collections of clutching maps $\{\{h_{vw}\}_{\overline{c_v} \cap \overline{c_w} \neq \emptyset}\}$ with fixed values over 0-cells. When the latter are dissected over compatible triples, we obtain the local homotopical data stated above, the first part corresponding to the data arising from an triple of cells $\overline{c_{v_1}} \cap \overline{c_{v_2}} \cap \overline{c_{v_3}} = \overline{c_\sigma}$, while the residual data takes the form of relative extension classes over all $(n-1)$-cells with orientation. Conversely, given any choice of such local homotopical data, it follows from the compatibility conditions, together with lemma 4, that the homotopy classes induced by the projections $\pi_i(\mathbf{h}_{\sigma\sigma''})$ of the maps $\mathbf{h}_{\sigma\sigma''}$ into components, with $i = 1, 2, 3$, together with the extension classes $[h_{vw}]$, determine a homotopy class of collections of clutching maps with fixed values over 0-cells. $\square$

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[8]Here the sum denotes the glueing of homotopy classes of cellularly smooth maps constructed in lemma 4, while $\mathbf{h}_{\partial \overline{c_{\sigma'}}}$ is the map of prescribed values at 0-cells. The set of such extension classes is a torsor for $\pi_k(V_G, \mathbf{e})$, where $\mathbf{e} = (e, e, e)$ (cf. corollary 2).

Corollary 2. *For any given standard lattice gauge field* $\mathrm{PT}^0_\Gamma : \mathscr{P}^0_\Gamma \to G$, *the set of extended lattice gauge fields compatible with* PT^0_Γ *is a torsor for the group*

$$G_{\mathscr{C}} = \prod_{\overline{c_\sigma} \subset \overline{c_\tau}} \left(\prod_{\overline{c_{\sigma''}} \subseteq \overline{c_\sigma}} G^{\sigma''}_{\sigma\tau} \right)$$

where, if $c_{\sigma''} \in \mathscr{C}_k$, $G^{\sigma''}_{\sigma\tau} \cong \pi_k(G, e)$. *Similarly, the set of cores of extended lattice gauge fields compatible with* PT^0_Γ *is a torsor for the subgroup* $H_{\min}$ *of the group*

$$H_{\mathscr{C}} = \left(\prod_{c_\sigma \in \mathscr{C}_{n-2}} \left(\prod_{\overline{c_{\sigma''}} \subseteq \overline{c_\sigma}} H^{\sigma''}_\sigma \right) \right) \times \left(\prod_{c_v, c_w \in \mathscr{C}_n, \, \overline{c_v} \cap \overline{c_w} \neq \emptyset} H_{vw} \right)$$

where, if $c_{\sigma''} \in \mathscr{C}_k$, $k = 1, \ldots, n-2$, $H^{\sigma''}_\sigma \cong \pi_k(V_G, \boldsymbol{e})$, $H_{vw} \cong \pi_{n-1}(G, e)$, *and consisting of elements whose components* $h_{vw} \in H_{vw}$, $h_{wv} \in H_{wv}$ *satisfy* $h_{wv} = h_{vw}^{-1}$.

Proof. Fix any extended lattice gauge field $\{\mathrm{PT}_{\mathscr{C}}\}$ (resp. its core $\{\mathrm{PT}_{\mathscr{C}}\}_{\min}$). Consider any other extended lattice gauge field $\{\mathrm{PT}'_{\mathscr{C}}\}$ coinciding with $\{\mathrm{PT}_{\mathscr{C}}\}$ over $\mathscr{P}^0_\Gamma$ (resp., such that its core coincides with $\{\mathrm{PT}_{\mathscr{C}}\}_{\min}$ over $\mathscr{P}^0_{\min}$). Then, for any flag $\overline{c_{\sigma''}} \subseteq \overline{c_\sigma} \subset \overline{c_\tau}$, the collection of products of representatives

$$g'_{\sigma\tau}|_{\overline{c_{\sigma''}}} \cdot g^{-1}_{\sigma\tau}|_{\overline{c_{\sigma''}}},$$

for the pair of extension classes $\left[g_{\sigma\tau}|_{\overline{c_{\sigma''}}} \right]$ and $\left[g'_{\sigma\tau}|_{\overline{c_{\sigma''}}} \right]$, coincides with a homotopy class of maps over $\overline{c_{\sigma''}}$, whose values at any 0-cell are equal to e, and whose values over $\partial \overline{c_{\sigma''}}$ are homotopically trivial, and hence, can be identified with an element in $\pi_k(G, e)$. A similar argument follows for the corresponding extension classes characterizing the cores, yielding for any flag $\overline{c_{\sigma''}} \subseteq \overline{c_\sigma}$ with $c_{\sigma''} \in \mathscr{C}_k$, $k = 1, \ldots, n-2$, an element in $\pi_k(V_G, \boldsymbol{e})$, and for every pair $c_v, c_w \in \mathscr{C}_n$ such that $\overline{c_v} \cap \overline{c_w} \neq \emptyset$, a pair of elements $h_{vw}, h_{wv} \in \pi_{n-1}(G, e)$ related as $h_{wv} = h_{vw}^{-1}$. $\square$

Remark 12. The set of standard lattice gauge fields for a given triangle-dual cell decomposition $\mathscr{C}$ acquires the structure of a group under pointwise-multiplication (isomorphic to G^{N_1}, where N_1 is the number of edges in the cellular network Γ) which we may denote as G^0_Γ, as a standard lattice gauge field is understood as a groupoid homomorphism $\mathrm{PT}^0_\Gamma : \mathscr{P}^0_\Gamma \to G$. Therefore, corollary 2 indicates that the set of extended lattice gauge fields, for a given triangle-dual cell decomposition $\mathscr{C}$, is a torsor for the group

$$G^0_\Gamma \times G_{\mathscr{C}},$$

and similarly, the set of possible cores of extended lattice gauge fields is a torsor for the group $G_{\min} \times H_{\min}$, where $G^0_{\min}$ is the subgroup of G^0_Γ consisting of groupoid homomorphisms $\mathrm{PT}^0_{\min} : \mathscr{P}^0_{\min} \to G$. Clearly, there is an injective homomorphism $G^0_{\min} \times H_{\min} \hookrightarrow G^0_\Gamma \times G_{\mathscr{C}}$.

Remark 13. The dissection of the core of an extended lattice gauge field in corollary 1, or what is the same, the relative homotopy classes of clutching maps, is strictly dependent on the fixed values over 0-cells of the latter, that is, the a priori choice of a collection of reference points

$$\bigsqcup_{c_\sigma \in \mathscr{C}_{n-2}} \left\{ \mathbf{h}_\sigma \left(c_{\sigma_0''} \right) \, : \, c_{\sigma_0''} \in \mathscr{C}_0, \sigma_0'' \supset \sigma \right\}.$$

Such is the basis for the contrast between an intrinsically *local* object (the extended lattice gauge fields), and the *global* notion of cellular bundle data representing an equivalence class of principal G-bundles: if a different choice of reference points is made, there would still exist a corresponding relative homotopy class of clutching maps defining an equivalent principal G-bundle, but the new extension classes resulting from the dissection procedure may turn out to be completely different than the previous ones. The interest in getting a better understanding of the correspondence between the extended lattice gauge fields and the equivalence classes of principal G-bundles is the motivation for the following definition.

Definition 12. Two extended lattice gauge fields $\{\mathrm{PT}_\mathscr{C}\}$, $\{\mathrm{PT}_\mathscr{C}'\}$ are said to be *cellularly equivalent* if their cores (that is, their induced relative homotopy classes of clutching maps $\{h_{vw}\}$, $\{h_{vw}'\}$) are cellularly equivalent.

Corollary 3. *Two extended lattice gauge fields on $\mathscr{C}$ are cellularly equivalent if and only if the principal G-bundles they determine are equivalent: cellular equivalence determines a fibration Φ from the set of extended lattice gauge fields onto the set of equivalence classes of principal G-bundles. The core of an extended lattice gauge field is the minimal local homotopy data (relative to $\mathscr{C}$) extending a standard lattice gauge field that is necessary to reconstruct a principal G-bundle $P \to M$, up to equivalence.*

Having understood the way a discretization of a gauge field may lead to recover the topology of a principal bundle, we can proceed and study the space of restricted gauge equivalence classes[9] of smooth connections $\mathscr{A}/\mathscr{G}_*$ over a given bundle P, up to equivalence. A fundamental by-product of theorem 1 and corollary 1 is the *cellular homotopy fibration*

$$\mathrm{PT}_\mathscr{C} \mapsto \{\mathrm{PT}_\mathscr{C}\}_{\min}$$

mapping a given cellular parallel transport map to the core of the extended lattice gauge field it induces. The cellular homotopy fibration then partitions a given space $\mathscr{A}/\mathscr{G}_*$ into equivalence classes

$$\mathscr{A}/\mathscr{G}_* = \bigsqcup_{\mathscr{K} \in \Phi^{-1}(\{P\})} (\mathscr{A}/\mathscr{G}_*)_{\mathscr{K}},$$

parametrized by the different cores $\mathscr{K} = \{\mathrm{PT}_\mathscr{C}\}_{\min}$ that define the bundle equivalence class supporting $\mathscr{A}/\mathscr{G}_*$.

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[9]That is, equivalence under gauge transformations whose value over the fibers of every base point in M is the identity. $\mathscr{G}_*$ is the space of such gauge transformations.

Definition 13. Let $\{P\}$ be the underlying equivalence class of principal G-bundles arising from a cellular parallel transport map $\mathrm{PT}_{\mathscr{C}}$, with induced space of connections modulo gauge transformations $\mathscr{A}/\mathscr{G}_*$, and $(\mathscr{A}/\mathscr{G}_*)_{\mathscr{K}} \subset \mathscr{A}/\mathscr{G}_*$ the equivalence class defined by the core $\mathscr{K} = \{\mathrm{PT}_{\mathscr{C}}\}_{\min}$. Two gauge equivalence classes of smooth connections in P are said to be $\mathscr{C}$-*equivalent*, or *equivalent at scale* $\mathscr{C}$, if (i) they lie in the same equivalence class $(\mathscr{A}/\mathscr{G}_*)_{\mathscr{K}}$, and (ii) their induced standard lattice gauge fields are the same. By a *microscopical deformation* of $\mathrm{PT}_{\mathscr{C}}$ in $\mathscr{A}/\mathscr{G}_*$, relative to a cellular network in $(M,\mathscr{C})$, we mean a smooth path in $(\mathscr{A}/\mathscr{G}_*)_{\mathscr{K}}$ preserving $\mathscr{C}$-equivalence, whose value at 0 corresponds to $\mathrm{PT}_{\mathscr{C}}$.

Altogether, we have constructed a series of fibrations that take the set of cellular parallel transport maps over $(M,\mathscr{C})$ as a starting point. Such fibrations could be understood diagramatically as follows:[10]

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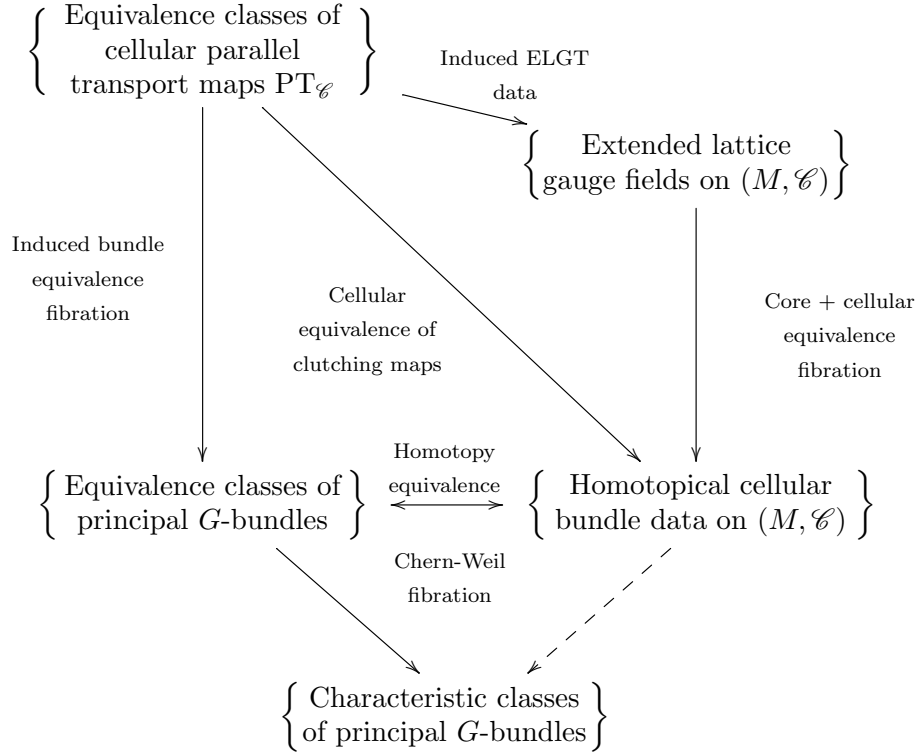
[10]So far, we have excluded the final topological fibration to the characteristic classes of a principal G-bundle in the present work. We plan to study the realization of characteristic classes from homotopical cellular bundle data, in the sense of the Chern-Weil theory, in a separate publication, as such problem is fundamental for its own sake.

A fundamental consequence of the torsor structure on the set of extended lattice gauge fields for a given pair $(M, \mathscr{C})$, described in corollary 2, and similarly, on the induced set of their cores, is the prescription of a collection of isomorphic subgroups of $G_{\min}^0 \times H_{\min} \subset G_\Gamma^0 \times G_\mathscr{C}$, parametrized by the space of isomorphism classes of principal G-bundles $\check{H}^1(M, \underline{G})$, namely, the subgroups $K_\mathscr{C}$ whose elements stabilize any given equivalence class $\{P\}$ of principal G-bundles (the isomorphism between any two stabilizers follows from the torsor structure on the space of extended lattice gauge fields). Consequently, there is a structure of a generalized homogeneous space[11] on $\check{H}^1(M, \underline{G})$,

$$\check{H}^1(M, \underline{G}) \cong \left(G_\Gamma^0 \times G_\mathscr{C} \right) / K_\mathscr{C},$$

with respect to the fibration on the right above, defined by taking cores and cellular equivalence on the set of extended lattice gauge fields, since there is an induced transitive action of $G_\Gamma^0 \times G_\mathscr{C}$ on $\check{H}^1(M, \underline{G})$. We consider that the previous transitive group actions deserve to be thoroughly studied, as they shed new light into the spaces of isomorphism classes of principal G-bundles on arbitrary manifolds, but the task seems to be far from trivial. Therefore, we plan to return to it in a subsequent publication.

6. Extended lattice gauge fields in small dimensions

To suplement section 5, we provide explicit homotopical characterizations of extended lattice gauge fields following from theorem 2 and corollary 1, for oriented manifolds of dimensions 2, 3 and 4. In fact, such characterizations suggest a recursive algorithm that could be implemented in the general case to determine a set of generators for the extension classes. It is important to keep in mind that for any Lie group G, the group $\pi_1(G, e)$ is abelian, $\pi_2(G, e)$ is always trivial, while $\pi_3(G, e)$ is always torsion-free, and hence isomorphic to $\mathbb{Z}^m$ for some m (see [12]).

• ($n = 2$; oriented surfaces) This is the simplest nontrivial case. The triangle-dual cellular decompositions are those for which at each vertex, exactly 3 edges merge (5.1). Particular examples are the tetrahedral, cubical and dodecahedral cellular representations of the 2-sphere. In such case, the only flags one needs to consider take the form $\overline{c_\tau} \subset \overline{c_v}$ with $c_\tau \in \mathscr{C}_1$, and an extended lattice gauge field is simply a sort of splitting of its core: the latter corresponds to (i) an assignment of a parallel transport to every path of the form $[\gamma_{\sigma v}]^{-1} \cdot [\gamma_{\sigma w}]$ for any pair of 2-cells $\overline{c_v}, \overline{c_w}$ sharing a common boundary $\overline{c_\tau}$ containing a 0-cell c_σ, and (ii) a collection of extension classes $[h_{vw}]$ relative to their boundary values, which group together as the points $\{\mathbf{h}_\sigma\} \subset V_G$. In turn, an extended lattice gauge field corresponds to (i) an assignment of a parallel transport to every path of the form $[\gamma_{\sigma \tau}]^{-1} \cdot [\gamma_{\sigma v}]$,[12] and (ii) a collection of extension classes $[g_{\tau v}]$ relative to their fixed (and

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[11]In the sense that the group acting transitively is not a Lie group, but the product of a discrete group and a Lie group.

[12]Observe that $[\gamma_{\sigma v}]^{-1} \cdot [\gamma_{\sigma w}] = \left([\gamma_{\sigma \tau}]^{-1} \cdot [\gamma_{\sigma v}] \right)^{-1} \cdot \left([\gamma_{\sigma \tau}]^{-1} \cdot [\gamma_{\sigma w}] \right)$.

compatible) boundary values $g_{\tau v}(p_\sigma)$, $g_{\tau v}(p_{\sigma'})$, $c_\sigma, c_{\sigma'} \subset \partial\overline{c_\tau}$. Hence, every class representative h_{vw} splits as $g_{\tau v}^{-1} \cdot g_{\tau w}$, for a pair of class representatives $g_{\tau v}$, $g_{\tau w}$. All of such extension classes can be parametrized by elements in $\pi_1(G, e)$ once auxiliary choices of extension maps are made.

FIGURE 3. Stereographic projection of a tetrahedral cellular decomposition of the 2-sphere, together with a cellular network. Base points are indicated with the letter p. The 2-cells are labelled with the subscripts v_i, the 1-cells with τ_j, and the 0-cells with σ_k.
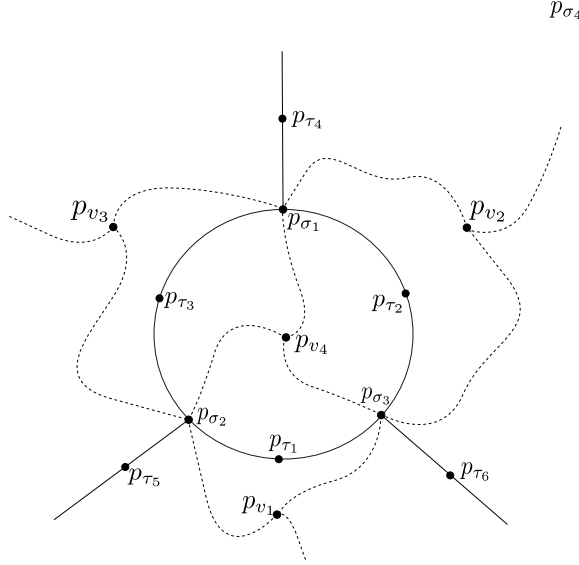
In fact, more can be said in dimension 2, regarding the projection of an extended lattice gauge field to its cellular bundle data. We can prescribe an equivalence class of principal G-bundles by means of a "canonical form", under cellular equivalence, of an extended lattice gauge field. Recall that the equivalence classes of principal G-bundles over an oriented surface S are parametrized by $\pi_1(G, e)$, and correspond to homotopy classes of transition functions for a trivialization $\{S \setminus \{p\}, \mathscr{U}\}$, where p is an arbitrary point in S and $\mathscr{U}$ is a small disk containing p, after a retraction from $(S \setminus \{p\}) \cap \mathscr{U}$ to a choice of some simply closed loop $\gamma \subset (S \setminus \{p\}) \cap \mathscr{U}$ is made. If we let γ be the boundary of a 2-cell $c_v \in \mathscr{C}$, $p = p_v$, and $\mathscr{U} = \mathscr{U}_v$, we can recover all equivalence classes by choosing any 1-cell $c_\tau \subset \partial\overline{c_v}$ (such that $\overline{c_\tau} = \overline{c_v} \cap \overline{c_w}$), choosing arbitrary elements in V_G for every $c_\sigma \in \mathscr{C}_0$, and decreeing all extension classes of clutching maps to be trivial, except for $[h_{vw}]$. The correspondence of such data and the equivalence classes of principal G-bundles then follows when the extension classes of clutching maps over the 1-subcells in $\partial\overline{c_v}$ are glued.

• ($n = 3$) Over an oriented 3-manifold, besides the prescription of a standard lattice gauge field, with generators corresponding to the parallel transports of all paths $\left[\gamma_{\sigma'_0\sigma}\right]$ joining the base point of a k-cell c_σ, $k = 1, 2, 3$, and a 0-cell $c_{\sigma'_0}$ in its boundary, the compatible values at 0-cells $c_{\sigma'_0}$ for the relative homotopy classes of glueing maps can be constructed as

$$g_{\sigma\tau}\left(p_{\sigma'_0}\right) = \mathrm{PT}^0\left(\left[\gamma_{\sigma'_0\sigma}\right] \cdot \left[\gamma_{\sigma'_0\tau}\right]\right).$$

Then, an extended lattice gauge field can be entirely described in terms of the flags of the form $\overline{c_\sigma} \subset \overline{c_\tau}$ and $\overline{c_\sigma} \subset \overline{c_v}$, with $c_\sigma \in \mathscr{C}_1$, $c_\tau \in \mathscr{C}_2$, $c_v \in \mathscr{C}_3$. This is so since the relative homotopy classes $[g_{\sigma\tau}]$, $[g_{\sigma v}]$ are determined by a single extension to $\overline{c_\sigma}$, which can be parametrized by elements in the group $\pi_1(G, e)$ after an auxiliary choice of extension maps is made. Then, the extensions of a class $[g_{\tau v}]$ over the boundary 1-subcell $\overline{c_\sigma}$ is determined recursively from the factorization property of representatives

$$g_{\tau v}|_{\overline{c_\sigma}} = g_{\sigma\tau}^{-1} \cdot g_{\sigma v}.$$

Then, the remaining extension class for $[g_{\tau v}]$ to the 2-cell c_τ is necessarily trivial, as a consequence of the triviality of $\pi_2(G, e)$.

• ($n = 4$) This case is also relatively easy to describe. Let $c_{\sigma'} \in \mathscr{C}_1$, $c_\sigma \in \mathscr{C}_2$, $c_\tau \in \mathscr{C}_3$, $c_v \in \mathscr{C}_4$. Once a standard lattice gauge field has been prescribed in terms of a set generators as in the case $n = 3$, and the induced choice of all compatible values at 0-cells for the relative homotopy classes of glueing maps is constructed, we proceed as follows. To construct the extension classes, the fundamental scaffolding is determined by the relative homotopy classes $[g_{\sigma'\sigma}]$, $[g_{\sigma'\tau}]$ and $[g_{\sigma'v}]$, which are determined by an extension class from $\partial\overline{c_{\sigma'}}$ to $\overline{c_{\sigma'}}$. As before, such extension classes can be parametrized by elements in $\pi_1(G, e)$, once an auxiliary choice of extension map is made. The homotopy classes $[g_{\sigma\tau}|_{\overline{c_{\sigma'}}}]$ and $[g_{\tau v}|_{\overline{c_{\sigma'}}}]$ are determined once again from the factorization property of representatives

$$g_{\sigma\tau}|_{\overline{c_{\sigma'}}} = g_{\sigma'\sigma}^{-1} \cdot g_{\sigma'\tau}, \qquad g_{\tau v}|_{\overline{c_{\sigma'}}} = g_{\sigma'\tau}^{-1} \cdot g_{\sigma'v},$$

and furthermore, their extensions from $\partial\overline{c_\sigma}$ to $\overline{c_\sigma}$ are trivial by the triviality of $\pi_2(G, e)$. To conclude, we only need to prescribe the extension of $[g_{\tau v}]$ from $\partial\overline{c_\tau}$ to $\overline{c_\tau}$, which can be parametrized by elements in $\pi_3(G, e)$ once an auxiliary choice of extension map is made.

A minimal example that holds in the previous cases, and in fact, for arbitrary dimensions, is the n-sphere, with a cellular decomposition induced from thinking about it as the boundary of an $(n+1)$-simplex or an $(n+1)$-cube. For instance, when $n = 2$, and S^2 corresponds to the boundary of a tetrahedron (figure 3), we can list a pair of paths for every 1-cell c_τ, joining p_τ and the 0-cells in its boundary $c_{\sigma_1}, c_{\sigma_2}$. Each of these paths gets assigned a corresponding parallel transport $\mathrm{PT}^0\left([\gamma_{\sigma_i\tau}]\right)$, playing the role of a prototypical lattice gauge field, in the usual sense. Now, the previous paths get complemented with the paths $\left[\gamma_{\sigma'_0\sigma}\right]$ in a cellular network, joining the

base point of an arbitrary k-cell c_σ, $k \geq 2$, to a 0-cell $c_{\sigma'_0}$ in its boundary, also getting assigned a parallel transport, and together fully determining a standard lattice gauge field. Finally, the prescription gets completed with an extension class of maps for every flag $\overline{c_\sigma} \subset \overline{c_\tau}$, and every k-subcell $\overline{c_{\sigma'}} \subseteq \overline{c_\sigma}$. When $d = 2$, there is one such extension class for every flag $\overline{c_\tau} \subset \overline{c_v}$ with $c_\tau \in \mathscr{C}_1$, and the extensions for the pairs $\overline{c_\tau} \subset \overline{c_v}$ and $\overline{c_\tau} \subset \overline{c_w}$, such that $\overline{c_\tau} = \overline{c_v} \cap \overline{c_w}$, are inverse to each other.

7. Pachner moves and independence of cellular decompositions

So far nothing has been said about the potential independence of our geometric and topological constructions under changes of the underlying triangle-dual cellular decomposition $\mathscr{C}$ that is required to be chosen. A rather special feature of the category of triangle-dual cellular decompositions on a manifold M, due to the simplicial nature of its objects, is the abundance of morphisms that allows us to connect and compare any given pair of them, up to dual P.L. equivalence, in a sistematic way. Namely, it was proved by Pachner [15] that any two smooth triangulations of a manifold are related by a sequence of the so-called *Pachner moves*. Let X^{n+1} be an abstract $(n + 1)$-simplex. By definition, we say that two different P.L. structures $\Delta : |K| \to M$ and $\Delta' : |K'| \to M$ on a manifold M differ by a Pachner move if there exists a pair of injective simplicial maps $\mu : L \to K$ and $\mu' : L' \to K'$, where

$$L = \bigcup_{l=0}^{k} X_l^n \subset \partial X^{n+1}, \qquad L' = \bigcup_{l=k+1}^{n+1} X_l^n \subset \partial X^{n+1},$$

for some arbitrary labeling $X_0^n, \ldots, X_{n+1}^n$ of the n-simplices in ∂X^{n+1} and $0 \leq k \leq n$, such that

(i) the map $(\Delta')^{-1} \circ \Delta : |K| \to |K'|$ is simplicial outside $|\mu(L \setminus \partial L)|$,
(ii) $\Delta(|\mu(L)|) = \Delta'(|\mu'(L')|)$,
(iii) $\Delta(|\mu(\sigma)|) = \Delta'(|\mu'(\sigma)|)$ for all $\sigma \in \partial L = \partial L'$.

Consequently, we are left to conclude that any two triangle-dual cellular decompositions $\mathscr{C}$ and $\mathscr{C}'$ of an n-manifold M are related by a corresponding sequence of dual Pachner moves – the corresponding transformations over the cell decompositions. Our first step towards cellular independence is to provide an elementary description of such transformations.

Remark 14. In the previous definition of a Pachner move, the $k+1$ different n-simplices of the simplicial complex L meet at a common $(n - k)$-face σ, while the $n - k + 1$ different n-simplices of the simplicial complex L' meet at a common k-face σ', dual to σ (in the case when $k = 0$, $L = \sigma$ and σ' is its complementary vertex v in X^{n+1}, and correspondingly for $k = n$). Therefore, when two triangulations Δ and Δ' of M differ by a Pachner move, such operation may be interpreted as the replacement of the $(n - k)$-face σ by the k-face σ' (unless $k = 0$, where the Pachner move consists of a refinement of an n-simplex through the extra interior vertex v, and

conversely for $k = n$), and there is a common refinement Δ'' containing exactly an additional 0-simplex v, corresponding to the intersection of σ and σ'. In turn, $\mathscr{C} \setminus \{c_\sigma\} \cong \mathscr{C}' \setminus \{c_{\sigma'}\}$, and the corresponding triangle-dual cellular decomposition $\mathscr{C}''$ contains an additional n-cell c_v than $\mathscr{C}$ and $\mathscr{C}'$ (figure 4). The n-cell closure $\overline{c_v}$ is topologically equivalent to the product $\overline{c_\sigma} \times \overline{c_{\sigma'}}$.

Lemma 5. *Any two triangle-dual cellular decompositions $\mathscr{C}$ and $\mathscr{C}'$ of M, arising from triangulations Δ and Δ', and related by a Pachner move as before, are degenerations of a smooth 1-parameter family of triangle-dual cellular decompositions $\{\mathscr{C}''_t\}_{t \in (0,1)}$,*

$$\lim_{t \to 0^+} \mathscr{C}''_t = \mathscr{C}, \qquad \lim_{t \to 1^-} \mathscr{C}''_t = \mathscr{C}',$$

with $\mathscr{C}''_t$ dual to Δ'', where Δ'' is the common refinement of Δ and Δ' described in remark 14.

Proof. Consider a choice of cellular decomposition $\mathscr{C}''$ dual to the common refinement Δ'', in such a way that $\mathscr{U}_u = \mathscr{U}_\sigma = \mathscr{U}_{\sigma'}$, and $\mathscr{C}''$ coincides with $\mathscr{C}$ and $\mathscr{C}'$ on $M \setminus \mathscr{U}_v$. Make $\mathscr{C}''$ correspond with $\mathscr{C}''_{1/2}$. Using the fact that $\overline{c_v} \cong \overline{c_\sigma} \times \overline{c_{\sigma'}}$, define a family on $(0, 1/2]$ by letting $\overline{c_v}^t$ degenerate to $\overline{c_\sigma}$, and on $[1/2, 1)$ by letting $\overline{c_v}$ degenerate to $\overline{c_{\sigma'}}$, in such a way that all cells outside $\mathscr{U}_v$ remain constant, all cells neighboring c_v^t transform into the corresponding cells neighboring c_σ, $c_{\sigma'}$, and the full family over $(0, 1)$ is smooth (figure 4). Observe that, in particular, in the case when $k = 0$ (resp. $k = n$), one of the two degenerations would not be present: topologically, the cellular decompositions for $t \in (0, 1]$ (resp. $t \in [0, 1)$) would be equivalent. $\square$

Remark 15. The triangle-dual cellular decomposition $\mathscr{C}_\Delta$ that the n-sphere acquires from its realization as the boundary of an $(n+1)$-simplex is actually a triangulation of S^n, and hence self-dual (that is, isomorphic to its own dual). An important consequence is the following geometric realization of the dual Pachner moves in $(S^n, \mathscr{C}_\Delta)$. Choose a k-cell c_σ in $\mathscr{C}_\Delta$. Then, there is a $(n-k)$-cell $c_{\sigma'}$ in $\mathscr{C}_\Delta$, complementary to the interior of the star of c_σ. Upon the choice of a smooth hemisphere H in S^n separating c_σ and $c_{\sigma'}$, we can identify each of the two components in $S^n \setminus H$ with the open sets $\mathscr{U}_\sigma$ and $\mathscr{U}_{\sigma'}$. In fact, more can be said about the topology of the cell degenerations: It is possible to foliate $S^n \setminus \{\overline{c_\sigma}, \overline{c_{\sigma'}}\}$ as a collection of cell subcomplexes, parametrized by $(0, 1)$, and each isomorphic to $\partial \overline{c_v}$. Such foliation can be extended to a foliation of the closed disk $\overline{\mathbb{D}^{n+1}}$, with the leave $\mathscr{L}_t$ corresponding to $\overline{c_v^t}$. This is the case since there exists a diffeomorphism

$$(0, 1) \times \overline{c_v} \cong \overline{\mathbb{D}^{n+1}} \setminus \{\overline{c_\sigma}, \overline{c_{\sigma'}}\}.$$

As an immediate corollary of Pachner's theorem and lemma 5, we obtain the following result, which is potentially useful to find algorithmically simple sequences of transformations between triangle-dual cellular decompositions.

FIGURE 4. (A) Pachner move on a triangulated surface, with
common refinement shown (center). (B) The corresponding
degenerations on the dual cellular decomposition.
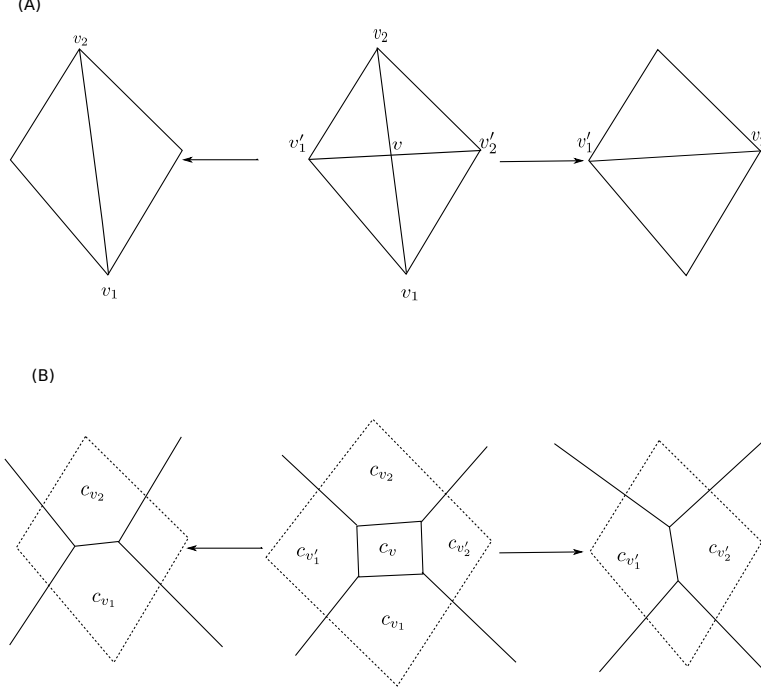
Corollary 4. *Any two triangle-dual cell decompositions of M are related by
a sequence of deformations and contractions of cells preserving the triangle-
dual property.*

Remark 16. In the same way that two cellular decompositions $\mathscr{C}$ and $\mathscr{C}'$
in a given manifold M, related by a dual Pachner move, can be thought of
as degenerations of a 1-parameter family of cellular decompositions $\{\mathscr{C}_t''\}$,
a pair of choices of path groupoids $\mathscr{P}_{\mathfrak{F}}$ and $\mathscr{P}_{\mathfrak{F}'}$, generated by complete
families $\mathfrak{F}$ and $\mathfrak{F}'$ adapted to $\mathscr{C}$ and $\mathscr{C}'$, may be understood as degenerations
of a 1-parameter family of path groupoids

$$\left\{ \mathscr{P}_{\mathfrak{F}_t''} \right\}_{t \in (0,1)}$$

resulting from a 1-parameter family of path families interpolating $\mathfrak{F}$ and $\mathfrak{F}'$,
and adapted to $\{\mathscr{C}_t''\}_{t \in (0,1)}$. The characteristic feature of such degenerations
follows from the isomorphism of cell complexes $\overline{c_v} = \overline{c_\sigma} \times \overline{c_{\sigma'}}$. Namely, for
any pair of subcells $c_{\sigma_0} \subset \overline{c_\sigma}$ and $c_{\sigma_0'} \subset \overline{c_{\sigma'}}$, the subcell of $\overline{c_v}$ whose closure
corresponds to $\overline{c_\sigma} \times \overline{c_{\sigma'}}$ would degenerate to c_{σ_0} or $c_{\sigma_0'}$. Such degeneration
implies the *fusion* of the local path families supported over the closures of
any two subcells in $\overline{c_v}$ collapsing to the same subcell in $\overline{c_\sigma}$ (or $\overline{c_{\sigma'}}$).

Consider a pair of triangle-dual cell decompositions $\mathscr{C}$ and $\mathscr{C}'$, related by a dual Pachner move as before, and moreover, assume they coincide over $M \setminus \mathscr{V}_\sigma = M \setminus \mathscr{V}_{\sigma'}$, where $\mathscr{V}_\sigma$ (resp. $\mathscr{V}_{\sigma'}$) denotes the open set in M given as the interior of the star of $\overline{c_\sigma}$ –the union of all cells in $\mathscr{C}$ (resp. $\mathscr{C}'$) whose closures intersect $\overline{c_\sigma}$ (resp. $\overline{c_{\sigma'}}$). Let us moreover assume that $\mathscr{C}$ and $\mathscr{C}'$ are equipped with choices of extended lattice gauge fields $\{\mathrm{PT}_\mathscr{C}\}$ and $\{\mathrm{PT}_{\mathscr{C}'}\}$ coinciding over all common cells in $M \setminus \mathscr{V}_\sigma = M \setminus \mathscr{V}_{\sigma'}$. Consider, as before, a family $\mathscr{C}''_t$ of triangle-dual cellular decompositions degenerating to $\mathscr{C}$ and $\mathscr{C}'$, and such that for all $t \in (0,1)$, the open set $\mathscr{V}_v$ –the interior of the star of $\overline{c_v^t}$– equals $\mathscr{V}_\sigma = \mathscr{V}_{\sigma'}$, and moreover, the restriction $\mathscr{C}''_t|_{M \setminus \mathscr{V}_v}$ coincides with the restrictions $\mathscr{C}|_{M \setminus \mathscr{V}_\sigma}$ and $\mathscr{C}'|_{M \setminus \mathscr{V}_{\sigma'}}$. Finally, equip such family with a smooth family of adapted path groupoids $\left\{ \mathscr{P}_{\mathfrak{F}''_t} \right\}$, and degenerating to $\mathscr{P}_{\mathfrak{F}}$ and $\mathscr{P}_{\mathfrak{F}'}$.

The next definition plays the role of a *generalized* cellular equivalence of extended lattice gauge fields, suited to consider smooth 1-parameter families of triangle-dual cellular decompositions arising from a dual Pachner move as before.

Definition 14. Two extended lattice gauge fields $\{\mathrm{PT}_\mathscr{C}\}$ and $\{\mathrm{PT}_{\mathscr{C}'}\}$ as before (that is, coinciding over all cells in $M \setminus \mathscr{V}_\sigma = M \setminus \mathscr{V}_{\sigma'}$) are called *local relatives* if there exists a smooth 1-parameter family

$$\left\{ \left\{ \mathrm{PT}_{\mathscr{C}''_t} \right\} \right\}_{t \in (0,1)},$$

whose restriction to $\mathscr{C}''_t|_{M \setminus \mathscr{V}_v}$ coincides with the respective restrictions of $\{\mathrm{PT}_\mathscr{C}\}$ and $\{\mathrm{PT}_{\mathscr{C}'}\}$ to $\mathscr{C}|_{M \setminus \mathscr{V}_\sigma}$ and $\mathscr{C}'|_{M \setminus \mathscr{V}_{\sigma'}}$, and such that

$$\lim_{t \to 0^+} \left\{ \mathrm{PT}_{\mathscr{C}''_t} \right\} = \{\mathrm{PT}_\mathscr{C}\}, \qquad \lim_{t \to 1^1} \left\{ \mathrm{PT}_{\mathscr{C}''_t} \right\} = \{\mathrm{PT}_{\mathscr{C}'}\}.$$

Theorem 3. *Let $\mathscr{C}$ and $\mathscr{C}'$ be two triangle-dual cellular decompositions of M, related by a dual Pachner move, together with a pair of extended lattice gauge fields $\{\mathrm{PT}_\mathscr{C}\}$ and $\{\mathrm{PT}_{\mathscr{C}'}\}$ such that the respective restrictions of their cores to $\mathscr{C}|_{M \setminus \mathscr{V}_\sigma}$ and $\mathscr{C}'|_{M \setminus \mathscr{V}_{\sigma'}}$ coincide. The principal G-bundles P and P' on M, induced by the cores $\{\mathrm{PT}_\mathscr{C}\}_{\min}$ and $\{\mathrm{PT}_{\mathscr{C}'}\}_{\min}$, are equivalent if and only if $\{\mathrm{PT}_\mathscr{C}\}$ and $\{\mathrm{PT}_{\mathscr{C}'}\}$ are local relatives.*

Proof. The proof is straightforward, and its essence is the fact that, when the cores of the extended lattice gauge fields are considered, or what is the same, the relative homotopy classes of collections of clutching maps, the notion of local relativity for a pair of extended lattice gauge fields is nothing but a (local) specialization of the notion of cellular equivalence, adapted to degenerating families of cell decompositions. The extra complications and subtleties implicit in the notion of (global) cellular equivalence, present in general, disappear, since the homotopy equivalence relation that defines it has been confined to the $(n-1)$-cells in $\overline{c_v}$ and the interior of its star.

$\square$

8. 't Hooft loop operators

The 't Hooft loop operator was originally formulated in the seminal work [20], in the context of quantum gauge theory, as an operator that could sense non-local aspects of gauge fields. Its significance is due to the fact that it provides a so-called "disorder parameter", and as such, it plays an essential role to study the phase diagram of quantum chromodynamics and in understanding the phenomenon of quark confinement. In this section we will study some of the topological and bundle theoretical aspects that are relevant in the mentioned construction (cf. [18]).

In its simplest form, the 't Hooft operation is defined on the space of smooth, G-valued holonomy maps over a space of thin homotopy classes of loops based at $x_0 \in M$, as in [3]. Let us consider an $(n-2)$-dimensional submanifold $L \subset M$[13] not containing x_0, and a smooth holonomy map Hol on (M, x_0). The 't Hoof loop operation T_L is then constructed by modifying Hol into a new holonomy map Hol^L on $M \setminus L$, according to the recipe

$$(8.1) \qquad \mathrm{Hol}^L(l) = \mathrm{T}_L[\mathrm{Hol}](l) = g^{N(L,l)}[\mathrm{Hol}](l),$$

where $g \in Z(G) \subset G$ is an element of the group's center that is arbitrary but fixed, and $N(L,l)$ denotes the linking number[14] between L and l (the linking number of these two different types of objects is well-defined, and in fact, a topological invariant, under isotopies of l in $M \setminus L$). The previous definition is, in fact, enough to determine the action of the operation in the space of smooth holonomy maps on $M \setminus L$.

The main goal of this section is to use our newly developed tools, to address the natural question of whether the principal G-bundles over $M \setminus L$, determined by $\mathrm{PT}_{\mathscr{C}}$ and $\mathrm{T}_L[\mathrm{PT}_{\mathscr{C}}]$, are equivalent. As we will see, such is the case, as a consequence of the connectedness of G, but in fact, while there is no longer a principal G-bundle structure over $L \subset M$, the defect of the cocycle condition over L turns out to be a manifestation of a more general topological structure, known as a *non-abelian bundle gerbe* [2, 4]. Thus, we are led to conclude that the natural setting to study the 't Hooft operation is not the category of principal G-bundles with connection, but instead a suitable category of non-abelian gerbes with the so-called "connective structures".

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[13]L will be called an "$(n-2)$-loop", or "$(n-2)$-knot", or "Dirac $(n-2)$-string", because in its original context, it lies inside a three-dimensional ("constant time") slice of a four-dimensional Lorentzian manifold.

[14]The linking numbers $N(L,l)$ can be calculated by means of the so-called *Seifert hypersurfaces* S, having L as boundary, as the intersection number of S and l, as the latter is independent of the choice of S, and which is defined due to the orientation in M. The existence of such hypersurfaces for any given 1-loop L is a classical result of Pontryagin-Frankl [6], and Seifert [19], in the cases $M = S^3$ or $M = \mathbb{R}^3$, where in fact the linking number can be prescribed in terms of the celebrated Gauss linking integral. The definiton of the linking number can be generalized analogously to dimensions higher than three [17]. Moreover, the linking number is invariant under thin homotopies, and descends to the group of based loops on (M, x_0), modulo thin homotopy, and hence, it can be understood algebraically as a homomorphism from such group to $\mathbb{Z}$.

Our starting point will be to extend equation (8.1) above, in order to define the 't Hooft operator in terms of smooth cellular parallel transport maps instead. Namely, let us consider a triangle-dual cell decomposition $\mathscr{C}$ on M, a collection of cell base points, and the corresponding path groupoid $\mathscr{P}_{\mathscr{C}}$. Recall that if a cellular network Γ is chosen, then from every cellular parallel transport map $\mathrm{PT}_{\mathscr{C}}$, we can extract the parallel transport data along the links contained in the discrete path subgroupoid $\mathscr{P}_{\Gamma}^0$, and that if a network parallel transport data is complemented with the homotopy cellular data, extracted from a minimal collection of path families $\mathfrak{F}_{\min}$, as in Section 4, we obtain sufficient information to characterize the bundle structure encoded in $\mathrm{PT}_{\mathscr{C}}$.

In order to define the 't Hooft operation on the space of smooth cellular parallel transport maps, we will henceforth consider a pair $(M, \mathscr{C})$, and also consider exclusively the $(n-2)$-loops L that fit in the $(n-2)$-skeleton of $\mathscr{C}$ (or conversely, if such a choice of $(n-2)$-loop $L \subset M$ is made, consider a triangle-dual cell decomposition $\mathscr{C}$ with the previous property). As opposed to the previous definition of the 't Hooft loop operation, we will now restrict to *fixed* Seifert hypersurfaces S for L, satisfying the following property: consider a hypersurface S_0 fitting in the $(n-1)$-skeleton of $\mathscr{C}$, such that $\partial S_0 = L$, and let S be a sufficiently small isotopy deformation of S_0, preserving the boundary constraint, such that no base point for $\mathscr{C}|_{M \setminus L}$ is contained in S, and such that there always exists a complete collection of path families $\mathfrak{F}$ whose intersection numbers with S are well-defined (remark 17), and coincide with the intersection numbers of S_0 for all the path families in $\mathfrak{F}$ not involving cells in S_0. We will say that S is *neighboring* S_0.

Let us now consider a loop l that admits a factorization as a product of paths

$$l = \gamma_r \cdot \cdots \cdot \gamma_1.$$

with $\gamma_1, \ldots, \gamma_r \in \mathscr{P}_{\mathscr{C}}$. Indeed, it is also possible to define intersection numbers $N(S, \gamma_i)$, provided that each γ_i intersects S transversally, in such a way that the usual linking number $N(L, l)$ can be recovered from them. In particular, the following formula holds

$$N(L, l) = N(S, l) = \sum_{i=1}^{r} N(S, \gamma_i).$$

Definition 15. For L an $(n-2)$-dimensional submanifold of M embedded in $\mathrm{Sk}_{n-2}(\mathscr{C})$, S a fixed choice of Seifert hypersurface for L (i.e. $\partial S = L$), neighboring some hypersurface S_0 embedded in $\mathrm{Sk}_{n-1}(\mathscr{C})$, and $g \in Z(G)$ a fixed element of the center of G, the 't Hoof operation $\mathrm{T}_{L,S}$ on a smooth cellular parallel transport map $\mathrm{PT}_{\mathscr{C}}$ on M is defined as the cellular parallel transport map $\mathrm{PT}_{\mathscr{C}}^{L,S}$ on $M \setminus L$ given by the formula

$$\mathrm{PT}_{\mathscr{C}}^{L,S}(\gamma) = \mathrm{T}_{L,S}\left[\mathrm{PT}_{\mathscr{C}}\right](\gamma) = g^{N(S,\gamma)}\left[\mathrm{PT}_{\mathscr{C}}\right](\gamma).$$

Observe that $\mathrm{T}_{L,S}$ is well defined in $\mathscr{P}_\mathscr{C}|_{M\setminus L}$, and when l is a loop in $\mathscr{P}_\mathscr{C}$, the parallel transports $\mathrm{PT}_\mathscr{C}^{L,S}(l)$ are independent of the choice of Seifert hypersurface $S \subset M$, and if, moreover, $l = \gamma_r \cdots \cdots \gamma_1$, then

$$\mathrm{PT}^L(l) = \mathrm{PT}^{L,S}(\gamma_r) \cdots \cdots \mathrm{PT}^{L,S}(\gamma_1),$$

for any Seifert hypersurface $S \subset M$.

Remark 17. The 't Hooft operation is, in principle, only defined for the parallel transport of elements $\mathscr{P}_\mathscr{C}$ that do not intersect L. However, if we now consider, for any pair $c_v, c_w \in \mathscr{C}_n$ such that $\overline{c_v} \cap \overline{c_w} \neq \emptyset$, the cellular path families $\mathscr{F}_{vw}$ defined in section 3, then, all paths γ_{vw}^x with x in the interior of $\overline{c_v} \cap \overline{c_w}$ would have a well-defined linking number $N\left(S, \gamma_{vw}^x\right)$. In such sense, we can define the linking number for *all* elements in the family $\mathscr{F}_{vw}$ (that is, *even* if x happens to lie in L), as an integral invariant of the intersection cell $\overline{c_\tau} = \overline{c_v} \cap \overline{c_w}$, whose value is equal to 0, if $\overline{c_\tau}$ is not contained in S, or 1, or -1, otherwise. We will denote such invariant as

$$\mathrm{Or}\left(S, c_{vw}\right).$$

Observe that, in particular, $\mathrm{Or}\left(S, c_{wv}\right) = -\mathrm{Or}\left(S, c_{vw}\right)$. The definition of the invariants $\mathrm{Or}\left(S, c_{vw}\right)$ is crucial, as the clutching maps were precisely defined in terms of the path families $\mathscr{F}_{vw}$. In particular, given a cellular parallel transport map $\mathrm{PT}_\mathscr{C}$, with induced clutching maps $\{h_{vw}(x) = \mathrm{PT}_\mathscr{C}\left(\gamma_{vw}^x\right)\}$, the 't Hooft operation induces the new collection of maps

$$\left\{ h_{vw}^{L,S} = g^{\mathrm{Or}(S, c_{vw})} h_{vw} : \overline{c_\tau} \to G \right\},$$

which will be the main object of interest. As we will prove in theorem 4, their restriction to $M \setminus L$ are clutching maps, thus determining a smooth principal bundle over $M \setminus L$. To understand the behavior of the collection $\left\{ h_{vw}^{L,S} \right\}$ over L, we require to recall the notion of non-abelian gerbe. Let (G, H, δ, α) be a cross module,[15] i.e., G and H be groups, together with a pair of homomorphisms

$$\delta : H \to G, \qquad \alpha : G \to \mathrm{Aut}(H),$$

and such that for all $g \in G$, $h, h' \in H$,

$$\delta\left(\alpha(g)(h)\right) = g\delta(h)g^{-1}, \qquad \alpha\left(\delta(h)\right)\left(h'\right) = h(h')h^{-1}.$$

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[15]In what follows, we will simply consider the case when $H = G$ is a connected Lie group, or $H = Z(G)$, the center of G, δ is the identity or the inclusion map, and α is the adjoint action.

Given an open cover $\mathfrak{U} = \{\mathscr{U}_v\}$ of M, a *non-abelian bundle gerbe*[16] [2, 4] is a collection of smooth maps

$$g_{uv} : \mathscr{U}_{uv} \to G, \qquad f_{uvw} : \mathscr{U}_{uvw} \to H,$$

satisfying

(8.2) $$g_{uv}|_{\mathscr{U}_{uvw}} \cdot g_{vw}|_{\mathscr{U}_{uvw}} = \delta(f_{uvw}) \cdot g_{uw}|_{\mathscr{U}_{uvw}},$$

and

(8.3) $$\alpha\left(g_{uv}|_{\mathscr{U}_{uvwt}}\right)\left(f_{vwt}|_{\mathscr{U}_{uvwt}}\right) \cdot f_{uvt}|_{\mathscr{U}_{uvwt}} = f_{uvw}|_{\mathscr{U}_{uvwt}} \cdot f_{uwt}|_{\mathscr{U}_{uvwt}}.$$

Theorem 4. *The collection of maps*

(8.4) $$\left\{h_{vw}^{L,S}\right\}_{\{c_v, c_w \in \mathscr{C}_n : \overline{c_v} \cap \overline{c_w} \neq \emptyset\}}$$

define a new collection of clutching maps for $(M \setminus L, \mathscr{C}|_{M\setminus L})$. The two collections of clutching maps are cellularly equivalent over $M \setminus L$,

$$\left[\left\{h_{vw}^{L,S}\right\}\right]\big|_{M\setminus L} = \left[\{h_{vw}\}\right]\big|_{M\setminus L},$$

thus defining the same equivalence class of principal G-bundles on $M \setminus L$. Moreover, the collection (8.4) determines clutching maps for a non-abelian gerbe on M.

Proof. As before, let us fix a Seifert surface S_0 for L contained in $\mathrm{Sk}_{n-1}(\mathscr{C})$, together with a neighboring Seifert surface S. Recall that in a triangle-dual celular decomposition $\mathscr{C}$, the closure of every $(n-2)$-cell c_σ is equal to the intersection of the closures of a triple of $(n-1)$-cells $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\}$. Since a map $h_{vw}^{L,S}$ differs from h_{vw} only when the $(n-1)$-cell closure $\overline{c_\tau} = \overline{c_v} \cap \overline{c_w}$ is contained in S_0, and an $(n-2)$-cell c_σ is contained in S_0 if and only if least one c_{τ_i} (but at most two) is contained in S_0, the cocycle condition will automatically hold over all $(n-2)$-cells not contained in S_0, the new clutching maps are identical to the original collection over $M \setminus S$, and the induced bundles would coincide there.

Assume now that the $(n-2)$-cell c_σ is contained in S_0. An $(n-2)$-cell in S_0 may either belong to $S_0 \setminus L$ or to L. In the first case, exactly two of the $(n-1)$-cells $\{c_{\tau_i}\}$ will be contained in S_0, while in the second case, exactly one of the $(n-1)$-cells $\{c_{\tau_i}\}$ will be contained in S_0.

Let us first consider a closed $(n-2)$-cell $\overline{c_\sigma}$ in $S_0 \setminus L$, corresponding to the intersection of S_0 and an $(n-1)$-cell $\overline{c_{\tau_1}}$ that is not in S_0. The remaining two $(n-1)$-cells $\overline{c_{\tau_2}}, \overline{c_{\tau_3}}$ that intersect at $\overline{c_\sigma}$ must belong to S_0, and their corresponding clutching maps would get modified. Due to the cyclic orientation that is given to $(n-1)$-cells incident in c_σ, the modification factor that is introduced by the 't Hooft operation in such pair of cells would appear

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[16]Originally, the notion of an (abelian) gerbe was introduced by Giraud in [7] (see [8] for a comprehensive and pedagogical introduction to the subject). The theory of gerbes has been vastly developed over the last decades, with applications spanning over a considerably broad variety of fields in Mathematics and Physics. Our approach to gerbes in this work is rather rudimentary, considering only what is strictly necessary for our purposes.

with oposite exponents. Thus, the cocycle condition would be preserved in such case. Hence, by restriction, we obtain a collection of clutching maps over $M \setminus L$, which determine a new principal G-bundle there.

It remains to prove that the collection of clutching maps on $M \setminus L$, induced by $\mathrm{T}_{L,S}$, is cellularly equivalent to the original collection. For this, we need to exhibit a homotopy between each pair of maps

$$\left\{ h_{vw}, h_{vw}^{L,S} \right\},$$

in such a way that the compatibility condition is satisfied at every $t \in [0,1]$. Consider any path $g(t)$ in the group with $g(0) = e$ and $g(1) = g$. The homotopy of pairs is defined in cases. When $\mathrm{Or}\,(S, c_{vw}) = 1$, we define

$$h_{vw}^{L,S}(t)(x) = g(t)h_{vw}(x),$$

and when $\mathrm{Or}(S, c_{vw}) = -1$, we define

$$h_{vw}^{L,S}(t)(x) = h_{vw}(x)g(t)^{-1}.$$

For the two of the three $(n-1)$-cells $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\}$, whose closures intersect in $\overline{c_\sigma}$ and belong to S_0, their associated clutching maps are modified. Now, according to the cyclic orientation, one of such maps is modified by a factor $g(t)$ multiplying on the left, while the other one is modified by a factor $g(t)^{-1}$ multiplying on the right. Thus, the modifications cancel, and the cocycle condition holds for every $t \in (0,1)$. and the two collections of clutching maps are cellularly equivalent on $M \setminus L$.

Finally, let us assume that c_σ belongs to L. Then, only one of the corresponding three cells $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\}$ would belong to S_0, say c_{τ_2}, and in such case, the only map from the triple that would be modified would be $h_{v_1 v_3}^{L,S}$. Thus, the new compatibility condition for the triple reads

$$h_{v_1 v_2}^{L,S} \cdot h_{v_2 v_3}^{L,S} = g^{\mathrm{Or}(S, c_{v_3 v_1})} h_{v_1 v_3}^{L,S},$$

hence, if we let

$$f_{v_1 v_2 v_3}^{L,S} = \begin{cases} g^{\mathrm{Or}(S, c_{v_3 v_1})} & \text{if} \quad c_\sigma \subset L, \\ \\ e & \text{otherwise,} \end{cases}$$

we conclude that the new structure that we have obtained from the 't Hooft operation is a collection of clutching maps for a non-abelian gerbe, with a $Z(G)$-twisting supported over L: the remaining compatibility condition (8.3) follows from the fact that the homomorphism α is trivial since $g \in Z(G)$, and moreover, an $(n-3)$-cell closure

$$\overline{c_u} \cap \overline{c_v} \cap \overline{c_w} \cap \overline{c_t}$$

has codimension 1 for the induced triangle-dual cell decomposition of L. Hence, the latter is the intersection of two $(n-2)$-cells in L, say $\overline{c_{uvw}}, \overline{c_{uwt}}$, and at the same time, the intersection of four $(n-2)$-cells $\overline{c_{uvw}}, \overline{c_{uvt}}, \overline{c_{uwt}}, \overline{c_{vwt}}$ in M, so that $c_{uvt}, c_{vwt} \not\subset L$. Therefore, exactly 2 of the corresponding four

maps $\left\{ f_{uvw}^{L,S}, f_{uvt}^{L,S}, f_{uwt}^{L,S}, f_{vwt}^{L,S} \right\}$ would be trivial, namely $f_{uvt}^{L,S}, f_{vwt}^{L,S}$, while the remaining two would provide canceling contributions in (8.3).

We could then talk about an induced isomorphism class of non-abelian gerbes

$$\left\{ P^{L,S} \right\},$$

induced by the 't Hooft operation in $\mathrm{PT}_{\mathscr{C}}$, and moreover, $\mathrm{PT}_{\mathscr{C}}^{L,S}$ should correspond to a gauge equivalence class of connective structures on $\left\{ P^{L,S} \right\}$ (but we will not dwell on the latter here). $\qquad\square$

Remark 18. The previously described non-abelian gerbe structure admits an analogous description in terms of glueing maps. Also, observe that, since the 't Hooft operation is trivial over parallel transport maps with values on a connected Lie group with trivial center, it can be understood in terms of the fibration $G \to G/Z(G)$ as a transformation on the space of non-abelian gerbes for G (containing in particular the space of Čech cocycles $\check{Z}^1\left(M,\underline{G}\right)$), fibering over the identity in $\check{Z}^1\left(M,\underline{G/Z(G)}\right)$.

Appendix A. Cellular decompositions on manifolds

Here we recall, for the sake of clarity and convenience, some standard notions on cell decompositions that we will require. We refer the reader to [11, 10] for further details.

Let M be an n-dimensional smooth manifold, assumed to be connected, not necessarily compact, and with or without boundary. A *smooth cellular decomposition* of M is defined inductively, as a disjoint union of subsets

$$M = \bigsqcup_{c_\sigma \in \mathscr{C}} c_\sigma,$$

together with continuous maps $\phi_\sigma^k : \overline{\mathbb{D}^k} \to M$ (where $\mathbb{D}^k$ denotes the unit disk in $\mathbb{R}^k$), such that each $\phi_\sigma^k|_{\mathbb{D}^k}$ is a diffeomorphism onto c_σ, in such a way that c_σ is a k-dimensional cell, and moreover, $\phi_\sigma^k\left(\partial\overline{\mathbb{D}^k}\right)$ is a disjoint union of cells of dimension at most $k-1$. We will denote an arbitrary smooth cell decomposition of M by $\mathscr{C} = \bigsqcup_{k=0}^{n} \mathscr{C}_k$, where $\mathscr{C}_k$ denotes the collection of k-dimensional cells in M. Similarly, the lth-skeleton of $\mathscr{C}$ is defined as $\mathrm{Sk}_l(\mathscr{C}) = \bigsqcup_{k=0}^{l} \mathscr{C}_k$.

A *flag* of length $m+1 \leq n+1$ in a cell decomposition $\mathscr{C}$ of M is a collection of nested cell closures $\overline{c_{\sigma_0}} \subset \overline{c_{\sigma_1}} \subset \cdots \subset \overline{c_{\sigma_m}}$ in M. A flag is said to be *gapless* if $\dim c_{\sigma_k} - \dim c_{\sigma_{k-1}} = 1$ for every $1 \leq k \leq m$. A flag is *complete* if its length is equal to $n+1$, so that $\dim c_{\sigma_k} = k$. A *descending flag* is defined in a similar way, by reversing the subset and cell dimension orderings.

In the case when M is orientable, and an orientation is chosen, there is a corresponding induced orientation in any cell $c_\sigma \in \mathscr{C}$, for any choice of a gapless flag starting at c_σ.

A barycentric subdivision $B(\mathscr{C})$ of $\mathscr{C}$ is any cellular decomposition of M satisfying the following property: there is a 1–1 correspondence between k-cells in $B(\mathscr{C})$ and gapless flags of length $k + 1$ in $\mathscr{C}$, in such a way that if $b^k_{\sigma_0...\sigma_k} \in B(\mathscr{C})_k$ corresponds to $\{\overline{c_{\sigma_0}} \subset \cdots \subset \overline{c_{\sigma_k}}\} \subset \mathscr{C}$, then $b^k_{\sigma_0...\sigma_k} \subset c_{\sigma_k}$ (this is possible since $\dim c_{\sigma_k} \geq k$ necessarily). Note how this definition is built inductively. In particular, there is exactly a 0-cell $b^0_\sigma \subset c_\sigma$ in $B(\mathscr{C})$ for every $c_\sigma \in \mathscr{C}$.

A smooth triangulation of M is a homeomorphism $\Delta : |K| \to M$, where K is an abstract simplicial complex and $|K|$ is its geometric realization, whose restriction to the interior of any simplex in $|K|$ is a diffeomorphism. We will denote by $\sigma, \tau, \ldots$ the elements in K (called the *faces* of K), and by $v, w, \ldots$ the vertices of it (in general, the maximal faces of an abstract simplicial complex K are called its *facets*, and K is said to be *pure* if all of its facets have the same dimension). Every triangulation has a canonically defined barycentric subdivision $B(\Delta)$.

Remark 19. The essential property that a cellular decomposition $\mathscr{C}$ of M must satisfy to qualify as "sufficiently good" for our purposes (and to avoid unnecessary patologies), is that for every k-cell c_σ ($k \geq 1$), $\partial\overline{c_\sigma}$ is is a piecewise-smooth $(k-1)$-sphere (c.f. definition 10). Let us call such a broader class of cellular decompositions *spherical*. Clearly, every triangle-dual cellular decomposition is spherical (a property inherited from the P.L structure), but the converse is not true: for instance, the hypercubical cellular decompositions of $\mathbb{R}^n$, determined by a lattice $\Lambda \subset \mathbb{R}^n$, are spherical but not triangle-dual, unless one consider the trivial case $n = 1$. It is thus natural to inquire what special properties single out the triangle-dual cellular decompositions among the former. An intuitively obvious property is their genericity, that is, "almost all" spherical cellular decompositions are triangle-dual.[17]

Lemma 6. *Triangle-dual cellular decompositions are stable under small smooth deformations. Moreover, every spherical cellular decomposition is a degeneration of a smooth family of triangle dual cellular decompositions.*

Proof. A cellular decomposition $\mathscr{C}_0$ would be unstable under small deformations if and only if there exists a family $\mathscr{C}_\epsilon$, $\epsilon \in \mathbb{R}$ of cellular decompositions which are combinatorially equivalent for small $\epsilon \neq 0$, and combinatorially inequivalent from $\mathscr{C}_0$ (figure 5). If that is the case, then there exist at least one family of k-cells c^ϵ_σ, for some $0 < k < n$, for which the limit of its closures $\overline{c^\epsilon_\sigma}$ collapses to a 0-cell c_0 in $\mathscr{C}_0$. In particular, all 0-cells in $\overline{c^\epsilon_\sigma}$ coalesce into c_0. Over the dual cell decompositions $\mathscr{C}^\vee_\epsilon$, each 0-cell in $\overline{c^\epsilon_\sigma}$ corresponds to an n-cell, and in the limit, all of these n-cells unify into the single n-cell $c^\vee_0$. It follows that the number of $(n-1)$-cells in $\partial\overline{c^\vee_0}$ would be larger that

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the number of $(n-1)$-cells in the boundary of any of the n-cells in $\mathscr{C}_\epsilon^\vee$ that are being unified into $\partial \overline{c_0^\vee}$.

If $\mathscr{C}$ is triangle dual, then in particular $\mathscr{C}^\vee$ would be a triangulation of M. By the previous argument, $\mathscr{C}$ cannot be a degeneration of a smooth family of combinatorially equivalent cell decompositions $\mathscr{C}_\epsilon$, otherwise, there would be a family of n-cells in $\mathscr{C}_\epsilon^\vee$ being unified into to the interior of an n-simplex in $\mathscr{C}^\vee$, for which the number of $(n-1)$-cells in its boundary (equal to n+1) is minimal, a contradiction.

Now, consider an arbitrary spherical cell decomposition $\mathscr{C}$ of M. Passing to the dual, it is always possible to construct a refinement of $\mathscr{C}^\vee$ which is triangulation Δ, and whose vertices coincide with the set of 0-cells in $\mathscr{C}^\vee$, that is $\mathrm{Sk}_0(\mathscr{C}^\vee) = \mathrm{Sk}_0(\Delta)$. It is then clear that, since the n-cells in $\Delta^\vee$ and $\mathscr{C}$ are in $1-1$ correspondence, it is possible to construct a family $\mathscr{C}_\epsilon$ of combinatorially equivalent cell decompositions in M, such that for every sufficiently small $\epsilon \neq 0$, $\mathscr{C}_\epsilon$ is equivalent to $\Delta^\vee$, so by construction $\mathscr{C}_\epsilon$ is triangle-dual, and such that the limit $\mathscr{C}_0$ coincides with $\mathscr{C}$. $\qquad\square$

FIGURE 5. (A) 1-cell contraction in a cellular decomposition $\mathscr{C}$ on a surface. (B) The effect over the dual decomposition $\mathscr{C}^\vee$.
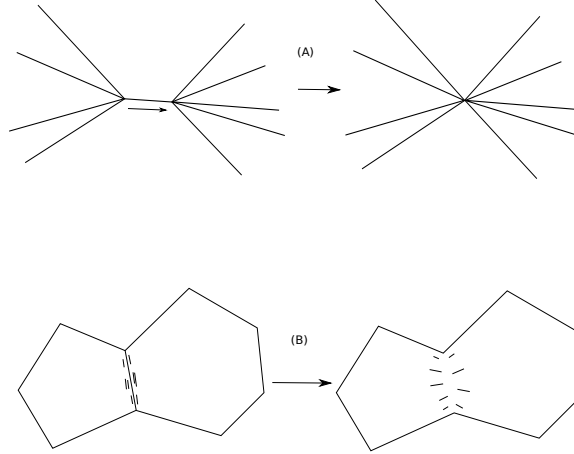
APPENDIX B. THE EQUIVALENCE THEOREM

Recall that a principal bundle P can be determined over an open cover $\mathfrak{U} = \{c_v\}_{v \in \mathscr{C}_n}$ by a Čech 1-cocycle, that is, a collection of smooth maps $g_{vw} : \mathscr{U}_{vw} \to G$ satisfying the cocycle conditions

(B.1) $\quad g_{wv} = g_{vw}^{-1} \quad$ on $\quad \mathscr{U}_{vw}, \qquad g_{uv}g_{vw}g_{wu} = e \quad$ on $\quad \mathscr{U}_{uvw},$

and any two such cocycles $\{g_{vw}\}$ and $\{g'_{vw}\}$ are equivalent if there exist a collection of smooth maps $\{g_v : \mathscr{U}_v \to G\}_{c_v \in \mathscr{C}_n}$ (*local gauge transformations*)

such that, over any $\mathscr{U}_{vw}$,

(B.2) $$g'_{vw} = g_v g_{vw} g_w^{-1}.$$

The cocycle condition is a statement on the 2-skeleton of the nerve $N(\mathfrak{U})$. In that sense, it will turn out to be very convenient to work with triangle-dual cellular decompositions and their associated open covers.

Theorem 5. *Let M be an oriented n-manifold, $n \geq 2$, together with a triangle-dual cellular decomposition $\mathscr{C}$ and a compatible star-like cover $\mathfrak{U}$ (see remark 1). There is a bijective correspondence*

$$\left\{ \begin{array}{c} \text{Cellular bundle data} \\ \text{in } M, \text{ relative to } \mathscr{C} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Equivalence classes of principal} \\ G\text{-bundles } P \to M \text{ trivialized over } \mathfrak{U} \end{array} \right\}$$

Proof. One implication is easy. Let us first consider an equivalence class of bundles with trivialization as a Čech 1-cocycle $\{g_{vw}\}$ up to equivalence, as described above. For any $c_\tau \in \mathscr{C}_{n-1}$, $v, w \subset \tau$, the restrictions

$$h_{vw} := g_{vw}|_{\overline{c_\tau}}$$

determine a representative of a class of cellular bundle data. This is so since every open set $\mathscr{U}_v$ is contractible by definition, and hence any local gauge transformation $g_v : \mathscr{U}_v \to G$ is smoothly homotopic to the identity. Therefore, any equivalent cocycle $\{g'_{vw}\}$ would be homotopic to $\{g_{vw}\}$ trough a homotopy of cocycles, inducing a corresponding cellular equivalence of maps $\{h_{vw}(t)\}$, $t \in [0, 1]$.

Conversely, let us consider an arbitrary choice of cellular bundle data $\mathscr{D}$ over $\mathscr{C}$, and for each $c_\tau \in \mathscr{C}_{n-1}$, together with a choice of orientation, fix data representatives, in the form of maps $h_{vw} : \overline{c_\tau} \to G$ such that for any cyclically oriented triples over any $(n-2)$-cell closure in $\partial \overline{c_\tau}$ the restriction to the diagonal $\Delta(\overline{c_\sigma} \times \overline{c_\sigma} \times \overline{c_\sigma})$ of the induced triples of maps lies in V_G. Our goal is to construct maps $g_{vw} : \mathscr{U}_{vw} \to G$ such that for any cyclically oriented triple $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\} \in \mathscr{C}_{n-1}$ as before, and the corresponding open sets

$$\mathscr{U}_{\tau_1} = \mathscr{U}_{v_2} \cap \mathscr{U}_{v_3}, \qquad \mathscr{U}_{\tau_2} = \mathscr{U}_{v_3} \cap \mathscr{U}_{v_1}, \qquad \mathscr{U}_{\tau_3} = \mathscr{U}_{v_1} \cap \mathscr{U}_{v_2},$$

the cocycle conditions (B.1) are satisfied. The construction will be done in two steps.

For the first step, let us consider, for every $c_\tau \in \mathscr{C}_{n-1}$, the intersections $\mathscr{U}_\tau \cap \mathrm{Sk}_{n-1}(\mathscr{C})$ (figure 6), and their subsets

$$\mathscr{Z}_\tau = \mathscr{U}_\tau \cap \left(\bigcup_{\tau' \in \mathscr{C}_{n-1}^\tau} \overline{c_{\tau'}} \right),$$

where $\mathscr{C}_{n-1}^\tau = \{c_{\tau'} \in \mathscr{C}_{n-1} : \overline{c_\tau} \cap \overline{c_{\tau'}} = \overline{c_\sigma}, c_\sigma \in \mathscr{C}_{n-2}\}$. Each $\mathscr{Z}_\tau$ is topologically a cylinder for $\partial \overline{c_\tau}$ without boundary. We will extend each $h_{vw}|_{\partial \overline{c_\tau}}$ to the whole $\mathscr{Z}_\tau$ in such a what that the cocycle conditions are still satisfyied. Consider any triple $\{c_{v_1}, c_{v_2}, c_{v_3}\}$ as in (5.1). For each $i = 1, 2, 3$

FIGURE 6. The intersection of $\mathscr{U}_\tau$ with the $(n-1)$-skeleton of $\mathscr{C}$.
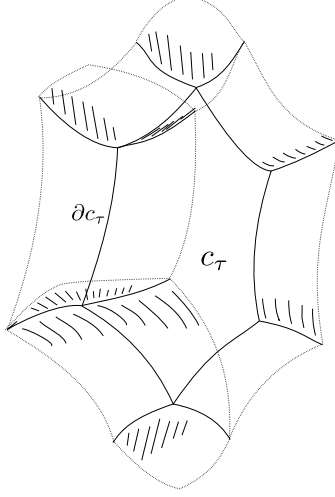
$j, k \neq i$, $(ijk) \sim (123)$, consider the intersection

$$\mathscr{I}_i = \mathrm{pr}_i^{-1}\left(h_{v_j v_k}(\overline{c_{\tau_i}} \cap \mathscr{U}_\sigma)\right) \cap V.$$

Now, for $i = 1, 2, 3$, and c_σ as above, consider any collection of piecewise-smooth maps $H_\sigma^i : (\overline{c_{\tau_1}} \cup \overline{c_{\tau_2}} \cup \overline{c_{\tau_3}}) \cap \mathscr{U}_\sigma \to G \times G \times G$, satisfying that

 (i) $H_\sigma^i\left((\overline{c_{\tau_1}} \cup \overline{c_{\tau_2}} \cup \overline{c_{\tau_3}}) \cap \mathscr{U}_\sigma\right) \subset \mathscr{I}_i$,
 (ii) $H_\sigma^i|_{\overline{c_\sigma}} = (h_{v_1 v_2}, h_{v_2 v_3}, h_{v_3 v_1})|_{\overline{c_\sigma}}$,
 (iii) If for $j \neq i$ there is $c_{\tau_j'}$ such that $\mathrm{int}\left(\overline{c_{\tau_j}} \cap \overline{c_{\tau_j'}}\right) \subset \mathscr{Z}_{\tau_i} \setminus \overline{c_{\tau_i}}$,[18] then

$$H_\sigma^i\Big|_{\mathrm{int}\left(\overline{c_{\tau_j}} \cap \overline{c_{\tau_j'}}\right)} = H_{\sigma'}^i\Big|_{\mathrm{int}\left(\overline{c_{\tau_j}} \cap \overline{c_{\tau_j'}}\right)}.$$

Such collections of maps would necessarily exist, since each one of the sets $(\overline{c_{\tau_1}} \cup \overline{c_{\tau_2}} \cup \overline{c_{\tau_3}}) \cap \mathscr{U}_\sigma$ deformation retracts to $\overline{c_\sigma}$. The set of maps $\{H_\sigma^i\}$ is parametrized by the elements $c_\sigma \in \mathscr{C}_{n-2}$, with a choice of cyclic orientation (5.1). Altogether, the functions $\{H_\sigma^i\}_{c_\sigma \in \mathscr{C}_{n-2}}$ determine extensions of each $h_{v_j v_k}$ to $\mathscr{Z}_{\tau_i}$, $(ijk) \sim (123)$. The extension to a subset $\mathscr{U}_{\tau_i} \cap \overline{c_{\tau_j}}$ is defined by means of the function H_σ^j. Conditions (ii) and (iii) above ensures that the overall process gives a well-defined extension of $h_{v_j v_k}$ over $\mathscr{Z}_{\tau_i}$, and condition (i) ensures that the new functions would continue satisfying the cocycle condition (B.1). Let us continue denoting by $h_{v_j v_k}$ such extension, now as a function over $\mathscr{Z}_{\tau_i} \cup c_{\tau_i}$.

The second step of the construction is to extend each $h_{v_j v_k}$ to the whole $\mathscr{U}_{\tau_i}$, and relies in the following observation. For any $c_\sigma \in \mathscr{C}_{n-2}$, consider its

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[18]Here, $\mathrm{int}\left(\overline{c_{\tau_j}} \cap \overline{c_{\tau_j'}}\right)$ denotes the $(k-2)$-cell whose closure is $\overline{c_{\tau_j}} \cap \overline{c_{\tau_j'}}$.

triple $\{c_{\tau_1}, c_{\tau_2}, c_{\tau_3}\}$, which has been cyclically oriented. For $i = 1, 2, 3$, the sets

$$\mathscr{B}_{\tau_i} = \mathscr{U}_{\tau_i} \cap \left(c_{\tau_i} \cup c_{v_j} \cup c_{v_k} \right), \qquad i \neq j, k, \quad j \neq k,$$

which are open cylinders for c_{τ_i} (whose boundary points contain $\mathscr{Z}_{\tau_i}$), satisfy $\mathscr{B}_{\tau_1} \cap \mathscr{B}_{\tau_2} \cap \mathscr{B}_{\tau_3} = \emptyset$ (figure 6). The latter property implies that, if we consider an arbitrary piecewise-smooth extension of $h_{v_j v_k}$ to $\mathscr{B}_{\tau_i}$, for every c_{τ_i}, then, there is a unique way to further extend such maps to the sets $\mathscr{U}_{\tau_i} \setminus \overline{\mathscr{B}_{\tau_i}}$ by forcing the cocycle condition to be satisfied on every triple intersection $\mathscr{U}_\sigma$, obtaining maps defined over $\mathscr{U}_{\tau_i}$. Hence, we can determine a collection of maps $\{g_{vw} : \mathscr{U}_{vw} \to G\}$, which define the cocycle we are looking for. We emphasize that the set of transition functions that we have constructed depends, a priori, on the choice of representatives $h_{vw} : \overline{c_\tau} \to G$ for each $c_\tau \in \mathscr{C}_{n-1}$ with a given orientation, and the choice of extensions of such h_{vw} to the sets $\mathscr{Z}_\tau \cup \mathscr{B}_\tau$.

To conclude the proof, we must show that, if we start with another set of representatives for our choice of cellular bundle data, or if we choose different extensions of the h_τ, the new cocycle would be equivalent to the previous one. By the previous construction, any choice of extensions of any pair of representatives h_{vw} and h'_{vw} of a given class $[h_{vw}]$ would be homotopic through cellularly-smooth functions satisfying the cocycle conditions, hence the resulting homotopies $g_{vw}(t)$ between each g_{vw} and g'_{vw} would define transition functions for every $t \in [0, 1]$. Then, there is a well-defined principal G-bundle over $M \times [0, 1]$. It is a standard result, following the homotopy invariance properties of principal bundles [13], that the bundles resulting by restriction to the boundary are isomorphic. $\square$

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