# On centrally generically tame algebras over perfect fields

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February 24, 2015

**Abstract**

We show that the central generic tameness of finite-dimensional algebra $\Lambda$ over a (possibly finite) perfect field, is equivalent to its non almost sharp wildness. In this case: we give, for each natural number $d$, parametrizations of the indecomposable $\Lambda$-modules with central endolength $d$, modulo finite scalar extensions, over rational algebras. Moreover, we show that the central generic tameness of $\Lambda$ is equivalent to its semigeneric tameness, and that in this case, algebraic boundedness coincides with central finiteness for generic $\Lambda$-modules.

## 1 Introduction

Denote by $k$ a fixed ground field and let $\Lambda$ be a finite-dimensional $k$-algebra. Given a $\Lambda$-module $G$, recall that by definition the *endolength* $G$ is its length as a right $\mathrm{End}_\Lambda(M)^{op}$-module. The module $G$ is called *generic* if it is indecomposable, of infinite length as a $\Lambda$-module, but with finite endolength. The algebra $\Lambda$ is called *generically tame* if, for each $d \in \mathbb{N}$, there is only a finite number of isoclasses of generic $\Lambda$-modules with endolength $d$. This notion was introduced and examined by W. W. Crawley-Boevey in [11] and [12]. In this paper we continue our exploration of the notion of generic tameness for finite-dimensional algebras $\Lambda$ over perfect fields (see [2], [3], and [4]). Our main results, stated below, apply to generically tame finite-dimensional algebras $\Lambda$ over a perfect (possibly finite) field $k$. In order to state precisely and comment these results we need to recall and introduce some terminology in the following definitions.

2010 *Mathematics Subject Classification*: 16G60, 16G70, 16G20.

*Keywords and phrases*: differential tensor algebras, ditalgebras, central endolength, generic modules, central finiteness, algebraic boundedness, tame and wild algebras.

**Definition 1.1.** For any $k$-algebra $B$ and $M \in B$-Mod, denote by $E_M := \mathrm{End}_B(M)^{op}$ its endomorphism algebra. Then, $M$ admits a structure of $B$-$E_M$-bimodule. By definition, the *endolength* of $M$, denoted by $\mathrm{endol}\,(M)$, is the length of $M$ as a right $E_M$-module.

A module $M \in B$-Mod is called *pregeneric* iff $M$ is indecomposable, with finite endolength but with infinite dimension over the ground field $k$. The algebra $B$ is called *pregenerically tame* iff, for each natural number $d$, there are only finitely many isoclasses of pregeneric $B$-modules with endolength $d$.

**Definition 1.2.** With the preceding notation, given $M \in B$-Mod, write $D_M = E_M/\mathrm{rad}E_M$ and denote by $Z_M$ the center of $D_M$. We shall say that the $B$-module $M$ is *centrally finite* iff $D_M$ is a division ring and $[D_M : Z_M]$ is finite. In this case, $[D_M : Z_M] = c_M^2$, for some positive integer $c_M$. If $M$ is centrally finite, the *central endolength* of $M$ is the number $c\text{-endol}\,(M) = c_M \times \mathrm{endol}\,(M)$.

The algebra $B$ is called *centrally pregenerically tame*, if for each $d \in \mathbb{N}$ there is only a finite number of isoclasses of centrally finite pregeneric $B$-modules with central endolength $d$.

**Definition 1.3.** Again with the preceding notation, a pregeneric $B$-module $G$ is called *algebraically rigid* if, for any algebraic field extension $\mathbb{L}$ of $k$, the $B^{\mathbb{L}}$-module $G^{\mathbb{L}}$ is pregeneric.

We say that a pregeneric $B$-module $G$ is *algebraically bounded* iff there exists a finite field extension $\mathbb{F}$ of $k$ and a finite sequence of algebraically rigid pregeneric $B^{\mathbb{F}}$-modules $G_1, \ldots, G_n$ such that $G^{\mathbb{F}} \cong G_1 \oplus \cdots \oplus G_n$.

An algebra $B$ is called *semipregenerically tame* if for each $d \in \mathbb{N}$ there is only a finite number of isoclasses of algebraically bounded and centrally finite pregeneric $B$-modules with central endolength $d$.

If $B$ is a finite-dimensional algebra, the notion of pregeneric $B$-module coincides with the usual notion of generic $B$-module. Hence, in this case we will eliminate the term "pre" which appears in the preceding denominations.

In [4] we obtained for a finite-dimensional semigeneric algebra $\Lambda$ over a perfect field, parametrizations of the centrally finite algebraically bounded $\Lambda$-modules "up to a finite extension of the field $k$". In the following definition we give a couple of useful variations of the notion of wildness with a similar relaxation of the ground field.

**Definition 1.4.** If $A$ and $B$ are $k$-algebras, a $k$-functor $F : A\text{-Mod} \longrightarrow B\text{-Mod}$ is called *sharp* (resp. *endosharp*) iff $F$ preserves indecomposables (resp. endofinite indecomposables), isomorphism classes of indecomposables (resp. of endofinite indecomposables), and induces isomorphisms $D_M \cong D_{F(M)}$, for each indecomposable (resp. endofinite indecomposable) $A$-module $M$, see [15, 4.1].

An algebra $B$ over a field $k$ is called *sharply wild* (resp. *endosharply wild*) iff there is a $B$-$k\langle x, y\rangle$-bimodule $Z$, which is free of finite rank by the right and such that the functor $Z \otimes_{k\langle x,y\rangle} - : k\langle x, y\rangle\text{-Mod} \longrightarrow B\text{-Mod}$ is sharp (resp. endosharp).

An algebra $B$ over a field $k$ is called *almost sharply wild* (resp. *almost endosharply wild*) iff there is a finite field extension $\mathbb{F}$ of $k$ such that $B^{\mathbb{F}}$ is sharply (resp. endosharply) wild.

Our main results are the following. Since generically tame finite-dimensional $k$-algebras are centrally generically tame, they apply to finite-dimensional generically tame algebras over perfect fields.

**Theorem 1.5.** *Let $\Lambda$ be a finite-dimensional algebra over a perfect field, then $\Lambda$ is centrally generically tame iff $\Lambda$ is not almost sharply wild iff $\Lambda$ is not almost endosharply wild.*

The preceding result together with theorem (7.1) can be considered as a generalization of the celebrated Tame and Wild Theorem of Drozd (see [13] and [10]), to the perfect ground field case.

**Theorem 1.6.** *Assume that $\Lambda$ is a finite-dimensional algebra over a perfect field. Then, $\Lambda$ is centrally generically tame iff $\Lambda$ is semigenerically tame. Moreover, if $\Lambda$ is semigenerically tame and $G$ is a generic $\Lambda$-module, then $G$ is centrally finite iff $G$ is algebraically bounded.*

Given a finite-dimensional algebra $\Lambda$ over a perfect field $k$, it is clear that $\Lambda$ is centrally generically tame whenever it is generically tame, and that $\Lambda$ is semigenerically tame whenever it is centrally generically tame. We do not know whether the former implication can be reversed. The notions of central finiteness and algebraic boundedness were introduced in [15], where the equivalence $\Lambda$ semigenerically tame iff $\Lambda^{\mathbb{K}}$ is generically tame, where $\mathbb{K}$ is the algebraic closure of $k$, is established. We do not know if centrally finite generic modules coincide with generic modules in the general case of a generically tame $\Lambda$.

A well known conjecture by Crawley-Boevey (see [12, 7.2 and 7.4]) asserts (if we restrict it to our context of a finite-dimensional algebra $\Lambda$ over a perfect field) that $\Lambda$ is either generically tame or generically wild, and not both. It includes the weaker conjecture that $\Lambda$ can not be simultaneously generically tame and generically wild. We remark that this last weaker statement is equivalent to the following: Every generic module over a generically tame finite-dimensional algebra $\Lambda$ is centrally finite. Indeed, this is a consequence of the following fact pointed out by Crawley-Boevey in [12]: Given a generic $\Lambda$-module $G$, we have that $\mathrm{End}_\Lambda(G)$ is a PI ring iff $D_G$ is finite dimensional over its center; thus $G$ is centrally finite iff $\mathrm{End}_\Lambda(G)$ is a PI ring.

Once our theorem (1.6) is proved, we can look and compare the parametrizations given in [4, 1.7] and (7.1). The latter one is given over rational algebras, while the former one is given over polynomial algebras. We stress the fact that, even though these theorems are proved using matrix problems techniques, the proofs are quite different. The scheme of the proof of (7.1) is closer to the one followed for the proof of Drozd's theorem in [10] and [13].

The proofs of our main results for algebras rely on the theory of *differential tensor algebras* (*ditalgebras* for short) and reduction functors first developed by the Kiev School of representation theory of algebras. For the general background on ditalgebras and their module categories, we refer the readers systematically to [5]. We tried to give precise references for the basic terminology and ditalgebra constructions.

## 2 Central pregeneric tameness

In this section, we recall from [4] and [7] the notion of semipregeneric tameness for layered ditalgebras. We introduce the notion of central pregeneric tameness for layered ditalgebras. Then, we recall results from [4], with minor adaptations, which will be used later.

**Definition 2.1.** Let $\mathcal{A}$ be a layered ditalgebra, with layer $(R, W)$, see [5, §4]. Given $M \in \mathcal{A}$-Mod, denote by $E_M := \mathrm{End}_{\mathcal{A}}(M)^{op}$ its endomorphism algebra. Then, $M$ admits a structure of $R$-$E_M$-bimodule, where $m \cdot (f^0, f^1) = f^0(m)$, for $m \in M$ and $(f^0, f^1) \in E_M$. By definition, the *endolength* of $M$, denoted by $\mathrm{endol}\,(M)$, is the length of $M$ as a right $E_M$-module.

A module $M \in \mathcal{A}$-Mod is called *pregeneric* iff $M$ is indecomposable, with finite endolength but with infinite dimension over the ground field $k$. A layered ditalgebra $\mathcal{A}$ is called *pregenerically tame* iff, for each natural number $d$, there are only finitely many isoclasses of pregeneric $\mathcal{A}$-modules with endolength $d$.

**Definition 2.2.** Given a layered ditalgebra $\mathcal{A}$ and $M \in \mathcal{A}$-Mod, write $D_M = E_M/\mathrm{rad}E_M$ and denote by $Z_M$ the center of $D_M$. We shall say that the $\mathcal{A}$-module $M$ is *centrally finite* iff $D_M$ is a division ring and $[D_M : Z_M]$ is finite. In this case, $[D_M : Z_M] = c_M^2$, for some positive integer $c_M$. If $M$ is centrally finite, the *central endolength* of $M$ is the number $c\text{-endol}\,(M) = c_M \times \mathrm{endol}\,(M)$.

A layered ditalgebra $\mathcal{A}$ is called *centrally pregenerically tame*, if for each $d \in \mathbb{N}$ there is only a finite number of isoclasses of centrally finite pregeneric $\mathcal{A}$-modules with central endolength $d$.

Notice that every finite-dimensional indecomposable $M$ over a Roiter ditalgebra $\mathcal{A}$, with layer $(R, W)$ such that $W_1$ is a finitely generated $R$-$R$-bimodule, is centrally finite, see [5, 5.12].

**Definition 2.3.** Given a layered ditalgebra $\mathcal{A}$ and a pregeneric $\mathcal{A}$-module $G$, we say that $G$ is *algebraically rigid* if, for any algebraic field extension $\mathbb{L}$ of $k$, the $\mathcal{A}^{\mathbb{L}}$-module $G^{\mathbb{L}}$ is pregeneric.

We say that a pregeneric $\mathcal{A}$-module $G$ is *algebraically bounded* iff there exists a finite field extension $\mathbb{F}$ of $k$ and a finite sequence of algebraically rigid pregeneric $\mathcal{A}^{\mathbb{F}}$-modules $G_1, \ldots, G_n$ such that $G^{\mathbb{F}} \cong G_1 \oplus \cdots \oplus G_n$.

A layered ditalgebra $\mathcal{A}$ is called *semipregenerically tame* if for each $d \in \mathbb{N}$ there is only a finite number of isoclasses of algebraically bounded centrally finite pregeneric $\mathcal{A}$-modules with central endolength $d$.

In the following, we enumerate a series of lemmas which are adaptations of the statements [2, 2.2–2.7].

**Lemma 2.4.** *Assume that* $\xi : \mathcal{A} \longrightarrow \mathcal{A}'$ *is a morphism of layered ditalgebras and consider the functor* $F_\xi : \mathcal{A}'\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ *induced by restriction using the morphism* $\xi$*. For* $M \in \mathcal{A}'\text{-Mod}$*, we have* $\mathrm{endol}\,(F_\xi(M)) \leq \mathrm{endol}\,(M)$*. Moreover:*

1. *If* $F_\xi$ *is full and faithful, it preserves centrally finite modules and, for a centrally finite* $M \in \mathcal{A}'\text{-Mod}$*, we have* $c\text{-endol}\,(F_\xi(M)) = c\text{-endol}\,(M)$*;*

2. *If the morphism* $\xi^{\mathbb{L}} = \xi \otimes 1_{\mathbb{L}} : \mathcal{A}^{\mathbb{L}} \longrightarrow \mathcal{A}'^{\mathbb{L}}$ *induces a full and faithful functor* $F_{\xi^{\mathbb{L}}} : \mathcal{A}'^{\mathbb{L}}\text{-Mod} \longrightarrow \mathcal{A}^{\mathbb{L}}\text{-Mod}$*, for any algebraic field extension* $\mathbb{L}$ *of* $k$*, then* $F_\xi$ *preserves pregeneric modules, algebraically rigid pregeneric modules, and algebraically bounded pregeneric modules. In this case, the ditalgebra* $\mathcal{A}'$ *is centrally pregenerically tame whenever* $\mathcal{A}$ *is so.*

3. *In addition to the assumptions of 2, suppose that* $\mathcal{A}$ *and* $\mathcal{A}'$ *are seminested, as in [5, 23.5]. Then,* $F_\xi$ *reflects pregeneric modules, algebraically rigid pregeneric modules, and algebraically bounded pregeneric modules.*

*Proof.* Item (1) and the first statement of (2) belong to [4, 2.6]. Item (3) admits essentially the same proof that [4, 2.6(3)], where we use that seminested ditalgebras are always Roiter ditalgebras (hence [5, 29.4] can still be applied). □

**Reminder 2.5.** Following [5], given a ditalgebra $\mathcal{A} = (T, \delta)$, we denote with a roman $A$ the subalgebra $[T]_0$ of degree zero elements of the underlying graded algebra $T$ of $\mathcal{A}$, see [5, §1]. Then, the categories $A$-Mod and $\mathcal{A}$-Mod share the same class of objects, but there are more morphisms in $\mathcal{A}$-Mod. There is a canonical embedding $L_{\mathcal{A}} : A\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ which is the identity on objects and $L_{\mathcal{A}}(f^0) = (f^0, 0)$ for any $f^0 \in \mathrm{Hom}_A(M, N)$.

We recall some terminology from [5] and [1]. Let $\mathcal{A} = (T, \delta)$ be any ditalgebra with layer $(R, W)$. Assume we have an $R$-$R$-bimodule decomposition $W_0 = W_0' \oplus W_0''$ with $\delta(W_0') = 0$. Consider the subalgebra $T'$ of $T$ generated by $R$ and $W' = W_0'$, and the subalgebra $B$ of $A$ generated by $R$ and $W_0'$. Then, the differential $\delta$ on $T$ restricts to a differential $\delta'$ on the algebra $T'$ and we obtain a new ditalgebra $\mathcal{B} = (T', \delta')$ with layer $(R, W')$. A layered ditalgebra $\mathcal{B}$ is called a *proper subalgebra of* $\mathcal{A}$ if it is obtained from an $R$-$R$-bimodule decomposition of $W_0$ as above. The ditalgebra $\mathcal{B}$ is essentially an algebra, and the module categories $\mathcal{B}$-Mod and $B$-Mod are canonically identified through the functor $L_{\mathcal{B}} : B\text{-Mod} \longrightarrow \mathcal{B}\text{-Mod}$.

A proper subalgebra $\mathcal{B}$ of a triangular ditalgebra $\mathcal{A}$ is called *initial* when $W_0'$ coincides with one of the terms of the triangular filtration of $W_0$, see [5, 14.8].

When $\mathcal{B}$ is a proper subalgebra of $\mathcal{A}$, the projection $\pi : T \longrightarrow T'$ yields a morphism of ditalgebras $\pi : \mathcal{A} \longrightarrow \mathcal{B}$, hence an *extension functor* $E := F_\pi : \mathcal{B}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$.

**Lemma 2.6** ([4](2.8))**.** *Assume that* $\mathcal{B}$ *is a proper subalgebra of the layered ditalgebra* $\mathcal{A}$ *and consider the extension functor* $E : \mathcal{B}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$*. Then,*

1. *The functor* $E$ *preserves isoclasses and indecomposables. Moreover, for any* $M \in \mathcal{B}\text{-Mod}$*, we have* $\mathrm{endol}\,(E(M)) = \mathrm{endol}\,(M)$*.*

2. *The functor $E$ preserves pregeneric modules, algebraically rigid pregeneric modules, and algebraically bounded pregeneric modules.*

3. *If $\mathcal{A}$ is a Roiter ditalgebra, then $M \in \mathcal{B}$-Mod is centrally finite if and only if $E(M) \in \mathcal{A}$-Mod is so and, in this case, $c$-endol$\,(E(M)) = c$-endol$\,(M)$.*

**Remark 2.7.** Given a seminested ditalgebra $\mathcal{A}$ over a field $k$, we shall consider the five basic operations $\mathcal{A} \mapsto \mathcal{A}^z$, where $z \in \{d, a, r, e, u\}$, called *deletion of idempotents* as in [5, 23.14], *regularization of a solid arrow* as in [5, 23.15], *absorption of a loop* as in [5, 23.16], *reduction of an edge* as in [5, 23.18] and *unravelling of a loop* as in [5, 23.23], and their corresponding reduction functors $F^z : \mathcal{A}^z\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$. The functor $F_z$ is full and faithful (by [5, 8.17, 8.19, 8.20], for $z \in \{a, d, r\}$, and by [5, 13.5], for $z \in \{e, u\}$).

**Lemma 2.8.** *Let $\mathcal{A}$ be a seminested ditalgebra over a field $k$. Suppose that $\mathcal{A}^z$ is obtained from $\mathcal{A}$ by some basic operation of type $z \in \{a, r, d\}$. Then, $\mathcal{A}^z$ is a seminested ditalgebra and we have:*

1. *The functor $F_z$ preserves endolength, central endolength, pregeneric modules, algebraically rigid pregeneric modules, and algebraically bounded pregeneric modules. Thus, $\mathcal{A}^z$ is centrally pregenerically tame whenever $\mathcal{A}$ is so.*

2. *The functor $F_z$ reflects pregeneric modules, algebraically rigid pregeneric modules, and algebraically bounded pregeneric modules.*

*Proof.* (1) This belongs to [4, 2.9 and 2.10].

(2) In case $z = a$, $F_a$ is the identity functor and our claim is clear. For $z \in \{d, r\}$, we need to keep in mind that $\mathcal{A}^z$ is seminested, in order to apply (2.4)(3) to the canonical projection $\xi : \mathcal{A} \longrightarrow \mathcal{A}^z$, since $F_z = F_\xi$. □

**Lemma 2.9.** *Let $\mathcal{A}$ be a seminested ditalgebra over a perfect field $k$ with layer $(R, W)$. Let $\mathcal{A}^X$ be the layered ditalgebra obtained from $\mathcal{A}$ by reduction, using a complete triangular admissible $B$-module $X$, for some proper subalgebra $B$ of $\mathcal{A}$, and consider the associated full and faithful functor $F_X : \mathcal{A}^X\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$, see [5, 12.10]. Suppose that the layer $(S, W^X)$ of $\mathcal{A}^X$ is seminested and that, for each algebraic field extension $\mathbb{L}$ of $k$, the admissible $R^{\mathbb{L}}$-module $X^{\mathbb{L}}$ is complete. Then,*

1. *For all $N \in \mathcal{A}^X$-Mod, we have that $N$ is centrally finite iff $F_X(N)$ is so and, in this case,*

$$\begin{cases} c\text{-endol}\,(N) \leq c\text{-endol}\,(F_X(N)) \\ c\text{-endol}\,(F_X(N)) \leq \operatorname{rank} X_S \times c\text{-endol}\,(N) \end{cases}$$

2. *The functor $F_X$ preserves and reflects pregeneric modules, algebraically rigid pregeneric modules, and algebraically bounded pregeneric modules.*

*Proof.* Without the "$c$", statement (1) follows from [5, 25.7], taking $E = \mathrm{End}_{\mathcal{A}^X}(N)^{op}$.

Our assumption requiring that for each algebraic field extension $\mathbb{L}$ of $k$ we have that $\widehat{X} = X^{\mathbb{L}}$ is complete gives that $F_{\widehat{X}} : \mathcal{A}^{\mathbb{L}X^{\mathbb{L}}}\text{-Mod} \longrightarrow \mathcal{A}^{\mathbb{L}}\text{-Mod}$ is full and faithful.

The proof of (2) is similar to the proof of [4, 2.11]: we use [5, 20.9] to guarantee that $X$ extends to $\mathbb{L}$, see [5, 20.11] and the fact that seminested ditalgebras are Roiter ditalgebras in order permit the application of [5, 29.4]. □

**Remark 2.10.** The last lemma applies to the functor $F_z : \mathcal{A}^z\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$, when $\mathcal{A}$ is a seminested ditalgebra over a perfect field $k$ and $\mathcal{A}^z$ is obtained from $\mathcal{A}$ by some basic operation of type $z \in \{e, u\}$, see [5, 23.18] and [5, 23.23].

Let us recall some usual notation.

**Notation 2.11.** Given a finite-dimensional algebra $\Lambda$ over any field $k$, denote by $\mathcal{P}(\Lambda)$ the category of morphisms between projective $\Lambda$-modules. If we write $J := \mathrm{rad}\Lambda$, then $\mathcal{P}^1(\Lambda)$ denotes the full subcategory of $\mathcal{P}(\Lambda)$ whose objects are the morphisms $\alpha : P \longrightarrow Q$ with image contained in $JQ$, and $\mathcal{P}^2(\Lambda)$ denotes the full subcategory of $\mathcal{P}^1(\Lambda)$ whose objects are the morphisms $\alpha : P \longrightarrow Q$ with kernel contained in $JP$. If $\Lambda$ splits over its radical, we can consider the Drozd's ditalgebra $\mathcal{D} = \mathcal{D}^\Lambda$, as in [5, 19.1], and the usual equivalence functor $\Xi_\Lambda : \mathcal{D}\text{-Mod} \longrightarrow \mathcal{P}^1(\Lambda)$, see [5, 19.8].

**Lemma 2.12** ([4](2.14))**.** *Given a finite-dimensional algebra $\Lambda$, over any field $k$, which splits over its radical, consider the Drozd's ditalgebra $\mathcal{D} = \mathcal{D}^\Lambda$, the usual equivalence functor $\Xi_\Lambda : \mathcal{D}\text{-Mod} \longrightarrow \mathcal{P}^1(\Lambda)$, and the cokernel functor* Cok : $\mathcal{P}^1(\Lambda) \longrightarrow \Lambda\text{-Mod}$. *Assume that $N \in \mathcal{D}\text{-Mod}$ and $M \in \Lambda\text{-Mod}$ are such that $\Xi_\Lambda(N) \in \mathcal{P}^2(\Lambda)$ and $M \cong \mathrm{Cok}\Xi_\Lambda(N)$. Then,*

1. *The $\mathcal{D}$-module $N$ is centrally finite iff the $\Lambda$-module $M$ is so. In this case, we have the following inequalities:*

$$\begin{cases} c\text{-endol}\,(N) \leq (1 + \dim_k \Lambda) \times c\text{-endol}\,(M) \\ c\text{-endol}\,(M) \leq \dim_k \Lambda \times c\text{-endol}\,(N) \end{cases}$$

2. *If $k$ is perfect, the module $N$ is an algebraically rigid (resp. algebraically bounded) pregeneric $\mathcal{D}$-module iff the module $M$ is an algebraically rigid (resp. algebraically bounded) generic $\Lambda$-module.*

**Corollary 2.13.** *Let $\Lambda$ be a finite-dimensional algebra over a perfect field $k$. Then, the algebra $\Lambda$ is centrally generically tame iff its Drozd's ditalgebra $\mathcal{D}$ is centrally pregenerically tame.*

*Proof.* Similar to the proof of [2, 4.5], using (2.12) instead of [2, 4.4]. □

## 3 Constructibility and wildness

In this section we consider a special type of ditalgebras, which can be constructed from finite-dimensional algebras over perfect fields. We show some properties of their pregeneric modules which follow from the corresponding properties for generic modules over finite-dimensional algebras. Then, we consider the notions of sharp wildness and endosharp wildness for layered ditalgebras and we show some examples.

**Definition 3.1.** Let $\mathcal{A} = (T, \delta)$ be a triangular ditalgebra with layer $(R, W)$ over a field $k$. Assume that $W$ is finitely generated as an $R$-$R$-bimodule. Then, $\mathcal{A}$ is called *elementary* iff $R \cong k \times \cdots \times k$, a finite product of copies of the field $k$. The ditalgebra $\mathcal{A}$ is called *semielementary* iff $R \cong M_{n_1}(k) \times \cdots \times M_{n_t}(k)$, a finite product of matrix algebras over the field $k$.

**Remark 3.2.** Assume that $\Lambda$ is a finite-dimensional algebra over a perfect field $k$. Then, we have a splitting $\Lambda = S \bigoplus J$ over its radical $J$, and we can consider the corresponding layered ditalgebra $\mathcal{D}_\Lambda$, see [5, 19.1]. Since $k$ is perfect, there is a finite field extension $\mathbb{L}$ of $k$ such that $(\mathcal{D}_\Lambda)^{\mathbb{L}}$ is semielementary, then we can consider its basification $(\mathcal{D}_\Lambda)^{\mathbb{L}b}$, as in [2, 3.3], which is an elementary ditalgebra (in particular, a seminested ditalgebra).

**Definition 3.3.** Given a perfect field $k$ and a finite field extension $\mathbb{L}$ of $k$, a seminested $\mathbb{L}$-ditalgebra $\mathcal{A}$ is called $\mathbb{L}$-*constructible* iff for some finite-dimensional $k$-algebra $\Lambda$ we have that the scalar extension $\mathcal{D}^{\mathbb{L}}$ of the Drozd's ditalgebra $\mathcal{D} = \mathcal{D}_\Lambda$ is semielementary and there is a finite sequence of reductions

$$\mathcal{D}^{\mathbb{L}b} \longmapsto \mathcal{D}^{\mathbb{L}bz_1} \longmapsto \mathcal{D}^{\mathbb{L}bz_1z_2} \longmapsto \cdots \longmapsto \mathcal{D}^{\mathbb{L}bz_1\cdots z_t},$$

and there is an isomorphism of layered ditalgebras $\mathcal{D}^{\mathbb{L}bz_1\cdots z_t} \cong \mathcal{A}$, for some finite set of reductions $\mathcal{D}^{\mathbb{L}bz_1\cdots z_{i-1}} \longmapsto \mathcal{D}^{\mathbb{L}bz_1\cdots z_i}$ of either of the types: absorption of a loop as in [5, 23.16], deletion of idempotents as in [5, 23.14], regularization as in [5, 23.15], edge reduction as in [5, 23.18], and unravelling as in [5, 23.23]. In this case, we also say that $\mathcal{A}$ *is* $\mathbb{L}$-*constructible from* $\Lambda$.

**Remark 3.4.** If a seminested ditalgebra $\mathcal{A}$ is $\mathbb{L}$-constructible (from some finite-dimensional $k$-algebra $\Lambda$) over the perfect field $k$, then $\mathcal{A}^{\mathbb{F}}$ is $\mathbb{F}$-constructible (from the same $k$-algebra $\Lambda$) for any finite field extension $\mathbb{F}$ of $\mathbb{L}$. Indeed, we need to keep in mind the meaning of the basification of a semielementary ditalgebra $\mathcal{D}^{\Lambda\mathbb{L}}$ with layer $(R, W)$ such that $R \cong M_{n_1}(\mathbb{L}) \times \cdots \times M_{n_t}(\mathbb{L})$; here $F^b$ is defined as $F^X$, where $X$ is the direct sum $X = X_1 \oplus \cdots \oplus X_t$, where each $X_j$ is a representative of the simple $M_{n_j}(\mathbb{L})$-module, see [2, 3.3], hence $R^{\mathbb{F}} \cong M_{n_1}(\mathbb{F}) \times \cdots \times M_{n_t}(\mathbb{F})$, and $X^{\mathbb{F}} \cong X_1^{\mathbb{F}} \oplus \cdots \oplus X_t^{\mathbb{F}}$, where each $X_j^{\mathbb{F}}$ is a representative of the simple $M_{n_j}(\mathbb{F})$-module. Thus, the basification functor $F^b$ : $(\mathcal{D}^{\Lambda^{\mathbb{L}}})^b$-Mod$\longrightarrow\mathcal{D}^{\Lambda^{\mathbb{L}}}$-Mod induces when extended to $\mathbb{F}$ the basification functor $F^b : (\mathcal{D}^{\Lambda^{\mathbb{F}}})^b$-Mod$\longrightarrow\mathcal{D}^{\Lambda^{\mathbb{F}}}$-Mod (modulo an isomorphism of layered ditalgebras which can be considered as an identification, see [5, 20.11]).

**Theorem 3.5.** *Assume that $\mathcal{A}$ is an $\mathbb{L}$-constructible seminested ditalgebra over a perfect field $k$. Then, for any pregeneric $\mathcal{A}$-module $G$, the algebra* $\mathrm{End}_{\mathcal{A}}(G)$ *is local and has nilpotent radical.*

*Proof.* This proof is similar to the proof of [3, 2.6]. We give the details. From [12, 4.2 and 4.4], the generic $\Lambda$-modules have local endomorphism algebras with nilpotent radical. Adopt the notation of definition (3.3). From [5, 20.13], we can identify the $\mathbb{L}$-ditalgebra $(\mathcal{D}^{\Lambda})^{\mathbb{L}}$ with $\mathcal{D}^{\Lambda^{\mathbb{L}}}$. Then, we can proceed as in the proof of [3, 2.6], to show that any pregeneric $\mathcal{D}^{\mathbb{L}}$-module $G$, the algebra $\mathrm{End}_{(\mathcal{D}^{\Lambda})^{\mathbb{L}}}(G)$ is local and has nilpotent radical. From [2, 3.3], we have the corresponding statement for $(\mathcal{D}^{\mathbb{L}})^b$, where $\mathcal{D} := \mathcal{D}^{\Lambda}$.

Consider the isomorphism of layered ditalgebras $\xi : \mathcal{D}^{\mathbb{L}bz_1\cdots z_t} \longrightarrow \mathcal{A}$ as an identification and, for $i \in [1,t]$, consider the corresponding reduction functor $F_i : \mathcal{D}^{\mathbb{L}bz_1\cdots z_i}\text{-Mod} \longrightarrow \mathcal{D}^{\mathbb{L}bz_1\cdots z_{i-1}}\text{-Mod}$. Then, the composition

$$F_1F_2\cdots F_t : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{D}^{\mathbb{L}b}\text{-Mod}$$

is a full and faithful functor which preserves pregeneric modules. Indeed, this is the case for each one of the factors, for instance, by (2.8) and (2.10). □

**Proposition 3.6.** *Let $\mathcal{A}$ be an $\mathbb{L}$-constructible seminested ditalgebra over a perfect field $k$ and take any field extension $\mathbb{F}$ of $\mathbb{L}$. Assume that $M, N \in \mathcal{A}$-Mod satisfy that $M^{\mathbb{F}}$ and $N^{\mathbb{F}}$ have a common non-zero direct summand. If $M$ is an endofinite indecomposable, then $M$ is a direct summand of $N$ in $\mathcal{A}$-Mod.*

*Proof.* This proof is similar to the proof of [3, 3.4]: first notice that Lemmas [3, 3.2 and 3.3] hold (with almost the same proofs: we just have to replace the chosen $k$-basis for $Z_0$ by a finite generating set of the $R$-module $Z_0$) if we substitute in the hypothesis "almost admissible" by "seminested". Then, if $M$ has infinite dimension, we can apply (3.5); if $M$ is finite-dimensional, we can apply [5, 5.12]. □

**Definition 3.7.** Let $\mathcal{A}$ and $\mathcal{B}$ be layered ditalgebras over any field $k$. Then, a $k$-functor $F : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{B}\text{-Mod}$ is called *sharp* (resp. *endosharp*) iff $F$ preserves indecomposables (resp. endofinite indecomposables), isomorphism classes of indecomposables (resp. of endofinite indecomposables), and induces isomorphisms $D_M \cong D_{FM}$, for each indecomposable (resp. endofinite indecomposable) $\mathcal{A}$-module $M$.

A layered ditalgebra $\mathcal{A}$ over a field $k$ is *sharply wild* (resp. *endosharply wild*) iff there is an $A$-$k\langle x,y\rangle$-bimodule $Z$, which is free of finite rank by the right and such that the following composition functor is sharp (resp. endosharp)

$$k\langle x,y\rangle\text{-Mod} \xrightarrow{Z\otimes_{k\langle x,y\rangle} -} A\text{-Mod} \xrightarrow{L_{\mathcal{A}}} \mathcal{A}\text{-Mod}.$$

We say that the bimodule $Z$ *realizes the sharp wildness* (resp. *realizes the endosharp wildness*) of $\mathcal{A}$. A layered ditalgebra $\mathcal{A}$ over a field $k$ is called *almost sharply wild* (resp. *almost endosharply wild*) iff there is a finite field extension $\mathbb{F}$ of $k$ such that $\mathcal{A}^{\mathbb{F}}$ is sharply wild (resp. endosharply wild).

**Remark 3.8.** 1. Any composition of sharp (resp. endosharp) functors is sharp (resp. endosharp).

2. Every sharp functor $F : \mathcal{B}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ preserving endofinite indecomposables is endosharp. In particular, any full and faithful $k$-functor $F : \mathcal{B}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ preserving endofinite modules is endosharp. This is the case of every reduction functor $F_z : \mathcal{A}^z\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ of type $z \in \{a, r, d, e, u\}$, by (2.4), (2.7), (2.8), and (2.10).

**Lemma 3.9.** *Any sharply wild (or endosharply wild) layered ditalgebra $\mathcal{A}$ over a perfect field is not centrally pregenerically tame.*

*Proof.* This proof is essentially the same that the proof of [2, 2.9], since the functor constructed there is sharp and endosharp. □

**Lemma 3.10.** *Assume that $\mathcal{B}$ is a proper subalgebra of the Roiter ditalgebra $\mathcal{A}$. Then, the corresponding extension functor $E : \mathcal{B}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ is sharp and endosharp.*

*Proof.* This follows from [15, 4.4] and [4, 2.8]. □

**Lemma 3.11.** *Assume that the seminested ditalgebra $\mathcal{A}^z$ is obtained from the seminested ditalgebra $\mathcal{A}$ by a basic operation of type $z \in \{a, r, d, e, u\}$. Then, if $\mathcal{A}^z$ is almost sharply wild (resp. almost endosharply wild), so is $\mathcal{A}$.*

*Proof.* We show first that if $\mathcal{A}^z$ is sharply (resp. endosharply) wild, so is $\mathcal{A}$. Assume, $\mathcal{A}^z$ is sharply (resp. endosharply) wild, so there is an $A^z$-$k\langle x, y\rangle$-bimodule $Z$, which is free of finite rank as a right $k\langle x, y\rangle$-module, such that the following composition functor is sharp (resp. endosharp):

$$k\langle x, y\rangle\text{-Mod} \xrightarrow{Z \otimes_{k\langle x,y\rangle} -} A^z\text{-Mod} \xrightarrow{L_{\mathcal{A}^z}} \mathcal{A}^z\text{-Mod}.$$

For $z \in \{a, r, d\}$, the canonical morphism of ditalgebras $\phi : \mathcal{A} \longrightarrow \mathcal{A}^z$ induces by restriction the full and faithful functor $F_z = F_\phi : \mathcal{A}^z\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$. By [5, 5.12], $F_z$ preserves indecomposability and isomorphism classes. From [5, 22.7], we know that the following diagram commutes up to isomorphism

$$\begin{array}{ccccc}
k\langle x, y\rangle\text{-Mod} & \xrightarrow{Z \otimes_{k\langle x,y\rangle} -} & A^z\text{-Mod} & \xrightarrow{L_{\mathcal{A}^z}} & \mathcal{A}^z\text{-Mod} \\
\| & & \downarrow & & \downarrow F_\phi \\
k\langle x, y\rangle\text{-Mod} & \xrightarrow{F_\phi(Z) \otimes_{k\langle x,y\rangle} -} & A\text{-Mod} & \xrightarrow{L_{\mathcal{A}}} & \mathcal{A}\text{-Mod}.
\end{array}$$

Hence, the lower composition of functors is sharp (resp. endosharp) and we just have to notice that the $A$-$k\langle x, y\rangle$-bimodule $F_\phi(Z)$ is free of finite rank as a right $k\langle x, y\rangle$-module.

For $z \in \{e, u\}$, $\mathcal{A}^z$ is defined as a ditalgebra $\mathcal{A}^X$ obtained from $\mathcal{A}$ by reduction using a special type of complete $\mathcal{B}$-module $X$, for a suitable initial subalgebra $\mathcal{B}$ of $\mathcal{A}$. By [5, 13.5] the functor $F_z = F^X$ is full and faithful. Then, again from [5, 5.12], we know that $F_z$ is sharp and we proceed as before. We recall that $F^z(Z)$ is finitely generated projective as a right $k\langle x, y\rangle$-module and hence, from [5, 22.6], also free.

Now, assume that $\mathcal{A}^z$ is almost sharply wild (resp. almost endosharply wild), so there is a finite field extension $\mathbb{F}$ of $k$ such that $\mathcal{A}^{z\mathbb{F}}$ is sharply wild (resp. endosharply wild). Then, recall that from [5, 20.4, 20.5, 20.6, 20.11], we can identify $\mathcal{A}^{z\mathbb{F}}$ with $\mathcal{A}^{\mathbb{F}z}$. Then, the preceding statement applied to the seminested $\mathbb{F}$-ditalgebra $\mathcal{A}^{\mathbb{F}}$, gives that $\mathcal{A}^{\mathbb{F}}$ is sharply wild (resp. endosharply wild), and we are done. $\square$

**Proposition 3.12.** *Assume that we have non-scalar elements $f(x) \in k[x]$ and $g(y) \in k[y]$. We say that an element $r(x, y) \in k[x, y]_{f(x)g(y)}$ admits an $f(x)g(y)$-zero iff there exist $\lambda, \mu \in k$ with $f(\lambda) \neq 0$, $g(\mu) \neq 0$, and $r(\lambda, \mu) = 0$. The following holds.*

1. *Any element $r(x, y) \in k[x, y]_{f(x)g(y)} \setminus \{0\}$ can be written as $r(x, y) = ur_0(x, y)$, where $r_0(x, y) \in k[x, y]$ has no irreducible factor in common with $f(x)g(y)$ and $u$ is an invertible element in $k[x, y]_{f(x)g(y)}$.*

2. *Any $r(x, y) \in k[x, y]_{f(x)g(y)}$ which admits an $f(x)g(y)$-zero must be a non-invertible element in $k[x, y]_{f(x)g(y)}$.*

3. *Given a non-zero $r(x, y) \in k[x, y]_{f(x)g(y)}$, there is either a finite field extension $\mathbb{F}_0$ of $k$ such that $r(x, y)$ is invertible in $\mathbb{F}[x, y]_{f(x)g(y)}$ for any finite field extension $\mathbb{F}$ of $\mathbb{F}_0$, or else there is a finite field extension $\mathbb{F}_1$ of $k$ such that $r(x, y) \in \mathbb{F}[x, y]_{f(x)g(y)}$ admits an $f(x)g(y)$-zero for any finite field extension $\mathbb{F}$ of $\mathbb{F}_1$.*

*Proof.* (1) and (2) are taken from [5, 24.4].

(3) Let $\mathbb{K}$ be an algebraic closure of $k$. If $r(x, y)$ is invertible in $\mathbb{K}[x, y]_{f(x)g(y)}$ then $r(x, y)s(x, y) = 1$, for some $s(x, y) \in \mathbb{K}[x, y]_{f(x)g(y)}$. Then, there is a finite field extension $\mathbb{F}_0$ of $k$ such that $s(x, y) \in \mathbb{F}_0[x, y]_{f(x)g(y)}$, and $r(x, y)$ is invertible in $\mathbb{F}[x, y]_{f(x)g(y)}$ for any finite field extension $\mathbb{F}$ of $\mathbb{F}_0$.

If $r(x, y)$ is not invertible in $\mathbb{K}[x, y]_{f(x)g(y)}$, we can apply (1) and write $r(x, y) = ur_0(x, y)$, where $r_0(x, y) \in \mathbb{K}[x, y]$ has no irreducible factor in common with $f(x)g(y)$ and $u$ is an invertible element in $\mathbb{K}[x, y]_{f(x)g(y)}$.

For any $h \in \mathbb{K}[x, y]$, we write $Z(h) := \{(\lambda, \mu) \in \mathbb{K}^2 \mid h(\lambda, \mu) = 0\}$. Then, from Bezout's theorem, we know that the set $Z(r_0(x, y)) \bigcap Z(f(x)g(y))$ is finite.

Given $(\lambda, \mu) \in \mathbb{K}^2$, it is clear that $f(\lambda) \neq 0$ and $g(\mu) \neq 0$ iff $(\lambda, \mu) \notin Z(f(x)g(y))$.

Since $\mathbb{K}$ is algebraically closed, $r_0(x, y)$ has infinitely many zeros $(\lambda, \mu) \in \mathbb{K}^2$, and we can choose such a zero $(\lambda, \mu)$ in $\mathbb{K}^2 \setminus [Z(r_0(x, y)) \cap Z(f(x)g(y))]$. Then, we can take the finite field extension $\mathbb{F}_1 = k(\lambda, \mu, c_1, \ldots, c_z)$ of $k$, where $c_1, \ldots, c_z \in \mathbb{K}$ are the algebraic elements present in $u$ and $r_0$. $\square$

**Theorem 3.13.** *Critical ditalgebras, as defined in [5, 24.5] over any field, are sharply wild and endosharply wild.*

*Proof.* For sharp wildness this was already noticed in the proof of [15, 4.6] for algebraically closed fields. The argument in the general case is the same. Namely, let $\mathcal{C}$ be a critical ditalgebra over a field $k$. Review carefully the development of [5, §24] and notice that it already contains the construction of a $C$-$k\langle x, y\rangle$-bimodule $B_0$ such that the functor $L_{\mathcal{C}}(B_0 \otimes_{k\langle x,y\rangle} -) : k\langle x, y\rangle\text{-Mod} \longrightarrow \mathcal{C}\text{-Mod}$ is sharp and endosharp: the functor that produces the wildness of the star algebra in [5, 30.2] is full and faithful, and for the extension functor involved we apply (3.10). Also, $B_0$ is free of finite rank as a right module, see [5, 22.7]. □

**Corollary 3.14.** *Assume that $\mathcal{A}$ is a seminested ditalgebra, over any field $k$, with differential $\delta$ and layer $(R, W)$. The following holds.*

1. *If $\delta(\alpha) = 0$ for some solid arrow $\alpha$ with either $Re_{s(\alpha)} \ncong k$ or $Re_{t(\alpha)} \ncong k$, then $\mathcal{A}$ is almost sharply wild and almost endosharply wild.*

2. *If $\delta(\alpha) = cv$, for some solid arrow $\alpha$, some dotted arrow $v$, and some non-zero element $c \in C := Re_{t(\alpha)} \otimes_k Re_{s(\alpha)}$, where $Re_{t(\alpha)} \ncong k$ and $Re_{s(\alpha)} \ncong k$, then either $\mathcal{A}$ is almost sharply wild and almost endosharply wild or there is a finite field extension $\mathbb{F}_0$ of $k$ such that for any finite field extension $\mathbb{F}$ of $\mathbb{F}_0$, the element $c \otimes 1$ is invertible in $C^{\mathbb{F}}$ and $\delta^{\mathbb{F}}(\alpha \otimes 1) = (c \otimes 1)(v \otimes 1)$.*

*Proof.* After deleting all the vertices of $\mathcal{A}$ different from $t(\alpha)$ and $s(\alpha)$, if necessary, and using (3.11), we can assume that $\mathcal{A}$ has only the points $t(\alpha)$ and $s(\alpha)$ (which may coincide). In case *1*, there is a finite field extension $\mathbb{F}$ of $k$ such that $\mathcal{A}^{\mathbb{F}}$ is a critical ditalgebra of one of the types (3) or (4) listed in [5, 24.5]. Thus, in this case $\mathcal{A}$ is almost sharply wild and almost endosharply wild. In case *2*, by (3.12), either there is a finite field extension $\mathbb{F}$ of $k$ such that $\mathcal{A}^{\mathbb{F}}$ is a critical ditalgebra of one of the types (1) or (2) listed in [5, 24.5], hence $\mathcal{A}$ is almost sharply wild and almost endosharply wild, or we are in the situation described in item *2*. □

# 4 Central endolength and scalar extension

In this section we recollect some facts on the behavior of the central endolength of centrally finite endofinite indecomposable modules over constructible ditalgebras, under finite field extension. They are obtained from the corresponding statements for finite-dimensional algebras proved in [15].

**Theorem 4.1.** *Let $\mathcal{A}$ be a seminested ditalgebra with layer $(R, W)$, over a perfect field $k$, and $M$ a finite-dimensional indecomposable $\mathcal{A}$-module. Then, there is a Galois field extension $\mathbb{F}_0$ of $k$ such that, for any Galois field extension $\mathbb{F}$ of $\mathbb{F}_0$, we have that the decomposition $M^{\mathbb{F}} \cong M_1 \oplus \cdots \oplus M_t$ of $M^{\mathbb{F}}$ as direct sum of indecomposables in $\mathcal{A}^{\mathbb{F}}$-mod satisfies that $D_{M_j} \cong \mathbb{F}$ and $\mathrm{endol}\,(M_j) = c\text{-endol}\,(M_j) = c\text{-endol}\,(M)$, for all $j \in [1, t]$. Moreover, if $e \in R$ is any idempotent, $e^{\mathbb{F}} = e \otimes 1$ is the induced idempotent in $R^{\mathbb{F}}$, and $j \in [1, t]$, we have $\ell_{E_{M_j}}(e^{\mathbb{F}} M_j) = c_M \times \ell_{E_M}(eM)$.*

*Proof.* This proof is similar to the proof of [4, 4.10(1)]. We need to have in mind that endomorphism algebras of finite-dimensional $\mathcal{A}$-modules are local finite-dimensional because $\mathcal{A}$ is seminested and we have [5, 5.13]. We also use that the canonical map $\alpha : (E_M)^{\mathbb{F}} \longrightarrow E_{M^{\mathbb{F}}}$ is an isomorphism of algebras, see [4, 4.12]. □

**Definition 4.2.** Assume that $\mathcal{A}$ is a seminested ditalgebra with layer $(R, W)$ and let $1 = \sum_{i=1}^{n} e_i$ be the decomposition of the unit of $R$ as a sum of centrally orthogonal primitive idempotents. Then, for $M \in \mathcal{A}$-Mod, the *support* of $M$ is the set of idempotents $e_i$ with $e_i M \neq 0$. The $\mathcal{A}$-module $M$ is called *sincere* iff $e_i M \neq 0$, for all $i \in [1, n]$.

Define the *endolength vector* of $M$ by $\underline{\ell}^e(M) = (\ell_1^e(M), \ldots, \ell_n^e(M))$, where $\ell_i^e(M) = \ell_{E_M}(e_i M)$, for $i \in [1, n]$. If $M$ is centrally finite, define the central endolength vector of $M$ by $c$-$\underline{\ell}^e(M) = c_M \times \underline{\ell}^e(M)$. Thus the endolength and the central endolength of $M$ are, respectively, $\mathrm{endol}\,(M) = \sum_{i=1}^{n} \ell_i^e(M)$ and $c$-$\mathrm{endol}\,(M) = \sum_{i=1}^{n} c$-$\ell_i^e(M)$.

**Corollary 4.3.** *Assume that $\mathcal{A}$ is a seminested ditalgebra over a perfect field $k$. Then, for any finite-dimensional indecomposable $\mathcal{A}$-module $M$ and any finite field extension $\mathbb{L}$ of $k$, in the decomposition $M^{\mathbb{L}} \cong M_1 \oplus \cdots \oplus M_t$ of the $\mathcal{A}^{\mathbb{L}}$-module $M^{\mathbb{L}}$ as a direct sum of indecomposables, we have that $c$-$\mathrm{endol}\,(M) = c$-$\mathrm{endol}\,(M_j)$ and $c$-$\underline{\ell}^e(M) = c$-$\underline{\ell}^e(M_j)$, for all $j \in [1, t]$.*

*Proof.* This proof is similar to the proof of [4, 4.11], where we use (4.1) instead of [4, 4.10]. Similarly, the equality on the central endolength vectors follows from (4.1) when applied to the idempotents $e_1, \ldots, e_n$ as in the last definition, first for a Galois field extension of $\mathbb{F}$ of $\mathbb{L}$ (hence of $k$), then to the Galois field extension $\mathbb{F}$ of $\mathbb{L}$, and then comparing the indecomposable modules appearing in both applications. □

**Proposition 4.4.** *Assume that $\mathcal{A}$ is an $\mathbb{L}$-constructible seminested ditalgebra over a perfect field $k$. Let $G$ be a pregeneric $\mathcal{A}$-module and $\mathbb{F}$ a finite field extension of $\mathbb{L}$. Then*

1. *$G^{\mathbb{F}} \cong m_1 G_1 \oplus \cdots \oplus m_t G_t$, where $m_1, \ldots, m_t \in \mathbb{N}$ and $G_1, \ldots, G_t$ are pairwise non-isomorphic pregeneric $\mathcal{A}^{\mathbb{F}}$-modules.*

2. *The module $G$ is centrally finite iff $G_j$ is centrally finite for some $j \in [1, t]$. In this case we also get that*

$$\dim_{Z_G}(D_G) = m_i^2 \times \dim_{Z_{G_i}}(D_{G_i}), \quad \textit{for each } i \in [1, t].$$

*Proof.* Since $\mathcal{A}$ is $\mathbb{L}$-constructible from a finite-dimensional $k$-algebra $\Lambda$, there is a functor $F : \mathcal{A}\text{-Mod} \longrightarrow (\mathcal{D}_\Lambda)^{\mathbb{L}b}\text{-Mod}$, given as a composition of functors of type $F^a$, $F^d$, $F^r$, $F^e$, $F^u$, $F_\xi$ associated to the basic operations: absorption of a loop, deletion of idempotents, regularization of an arrow, edge reduction, and unravelling, or replacement of a layered ditalgebra by an isomorphic one, see

(3.3) and (2.4). From [5, 20.4, 20.5, 20.6, 20.11, 20.12, and 20.13], we have the following diagram which commutes up to isomorphism,

$$\begin{array}{ccccccc} \mathcal{A}^{\mathbb{F}}\text{-Mod} & \xrightarrow{\widehat{F}} & (\mathcal{D}_{\Lambda^{\mathbb{F}}})^b\text{-Mod} & \xrightarrow{F^{\hat{b}}} & \mathcal{D}_{\Lambda^{\mathbb{F}}}\text{-Mod} & \xrightarrow{\mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}}} & \Lambda^{\mathbb{F}}\text{-Mod} \\ \Big\uparrow {\scriptstyle (-)^{\mathbb{F}}} & & & & & & \Big\uparrow {\scriptstyle (-)^{\mathbb{F}}} \\ \mathcal{A}\text{-Mod} & \xrightarrow{F} & (\mathcal{D}_{\Lambda^{\mathbb{L}}})^b\text{-Mod} & \xrightarrow{F^{b}} & \mathcal{D}_{\Lambda^{\mathbb{L}}}\text{-Mod} & \xrightarrow{\mathrm{Cok}\Xi_{\Lambda^{\mathbb{L}}}} & \Lambda^{\mathbb{L}}\text{-Mod} \end{array}$$

where $\widehat{F}$ is the composition of the functors induced by the functors appearing in the factorization of $F$: they are again of type $F^a$, $F^d$, $F^r$, $F^e$, $F^u$ or $F_\xi$, and $F^{\hat{b}}$ is the corresponding basification equivalence. They all preserve pregeneric modules. Moreover, $F_\xi$, $F^d$, and $F^r$ preserve central endolength, and $F^u$, $F^e$, $F^b$ behave towards central endolength as described in (2.9).

Let $G$ be a pregeneric $\mathcal{A}$-module, then $H := \mathrm{Cok}\Xi_{\Lambda^{\mathbb{L}}} F^b F(G)$ is a generic $\Lambda^{\mathbb{L}}$-module and, applying [15, 2.18(b) and 2.14(c)], we get $H^{\mathbb{F}} \cong m_1 H_1 \oplus \cdots \oplus m_t H_t$, for some $m_1, \ldots, m_t \in \mathbb{N}$ and some pairwise non-isomorphic pregeneric $\Lambda^{\mathbb{F}}$-modules $H_1, \ldots, H_t$.

Since $\mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}} F^{\hat{b}} \widehat{F}(G^{\mathbb{F}}) \cong \mathrm{Cok}\Xi_{\Lambda^{\mathbb{L}}} F^b F(G)^{\mathbb{F}} \cong H^{\mathbb{F}}$, using that the functor $\mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}} F^{\hat{b}} \widehat{F}$ reflects isomorphisms and [5, 29.4], we get $\mathcal{A}^{\mathbb{F}}$-modules $G_1, \ldots, G_t$ with $G^{\mathbb{F}} \cong m_1 G_1 \oplus \cdots \oplus m_t G_t$ and $\mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}} F^{\hat{b}} \widehat{F}(G_j) \cong H_j$, for each $j \in [1, t]$. Then, using that the functors $F^z$ reflect pregeneric modules, we know that each $G_j$ is pregeneric, and we have proved (1).

We also know from [15, 2.18(b)] that $H$ is centrally finite iff $H_j$ is so for some $j$. Then, using the fact that the functor $\mathrm{Cok}\Xi_{\Lambda^F} F^{\hat{b}} \widehat{F}$ is sharp, because it is a composition of sharp functors, we know that $G$ is centrally finite iff $H$ is so, iff $H_j$ is so, iff $G_j$ is so. The formula $\dim_{Z_H}(D_H) = m_i^2 \times \dim_{Z_{H_i}}(D_{H_i})$ for each $i$, established in the proof of [15, 2.18], gives the formula in (2) using again that the functor $\mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}} F^{\hat{b}} \widehat{F}$ is sharp. □

**Proposition 4.5.** *Assume that $\mathcal{A}$ is an $\mathbb{L}$-constructible seminested ditalgebra over a perfect field, let $\mathbb{F}$ a finite field extension of $\mathbb{L}$, and consider the corresponding scalar restriction functor $F_\xi : \mathcal{A}^{\mathbb{F}}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$. Let $H$ be a pregeneric $\mathcal{A}^{\mathbb{F}}$-module. Then*

1. *$F_\xi(H) \cong H_1 \oplus \cdots \oplus H_t$, where $H_1, \ldots, H_t$ are pregeneric $\mathcal{A}$-modules.*

2. *There is $i_0 \in [1, t]$ such that the module $H$ is a direct summand of $H_{i_0}^{\mathbb{F}}$.*

*Proof.* This proof is similar to the previous one, but we use scalar restrictions instead of scalar extensions. Namely, we consider the following diagram which, by [4, §3], commutes up to isomorphism

$$\begin{array}{ccccccc} \mathcal{A}^{\mathbb{F}}\text{-Mod} & \xrightarrow{\widehat{F}} & (\mathcal{D}_{\Lambda^{\mathbb{F}}})^b\text{-Mod} & \xrightarrow{F^{\hat{b}}} & \mathcal{D}_{\Lambda^{\mathbb{F}}}\text{-Mod} & \xrightarrow{\mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}}} & \Lambda^{\mathbb{F}}\text{-Mod} \\ \Big\downarrow {\scriptstyle F_\xi} & & & & & & \Big\downarrow {\scriptstyle F_{\xi_0}} \\ \mathcal{A}\text{-Mod} & \xrightarrow{F} & (\mathcal{D}_{\Lambda^{\mathbb{L}}})^b\text{-Mod} & \xrightarrow{F^{b}} & \mathcal{D}_{\Lambda^{\mathbb{L}}}\text{-Mod} & \xrightarrow{\mathrm{Cok}\Xi_{\Lambda^{\mathbb{L}}}} & \Lambda^{\mathbb{L}}\text{-Mod} \end{array}$$

where $F$ and $\widehat{F}$ are compositions of functors of type $F^z$, where $z \in \{a, r, d, e, u\}$ or induced by isomorphisms of layered ditalgebras, $F^b$ and $F^{\hat{b}}$ are the corresponding basification equivalences, and $F_{\xi_0}$ is the restriction functor. All the functors represented by horizontal arrows in the diagram preserve and reflect pregeneric modules.

Let $H$ be a pregeneric $\mathcal{A}^{\mathbb{F}}$-module, then $G := \mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}} F^{\hat{b}} \hat{F}(H)$ is a generic $\Lambda^{\mathbb{F}}$-module and, applying [15, 2.15], we get $F_{\xi_0}(G) \cong G_1 \oplus \cdots \oplus G_t$, for some generic $\Lambda^{\mathbb{L}}$-modules $G_1, \ldots, G_t$, and $G$ is a direct summand of $G_{i_0}^{\mathbb{F}}$ for some $i_0 \in [1, t]$.

Since $\mathrm{Cok}\Xi_{\Lambda^{\mathbb{L}}} F^b F F_{\xi}(H) \cong F_{\xi_0} \mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}} F^{\hat{b}} \widehat{F}(H) \cong F_{\xi_0}(G)$, using that the functor $\mathrm{Cok}\Xi_{\Lambda^{\mathbb{L}}} F^b F$ reflects isomorphisms and [5, 29.4], we get $\mathcal{A}$-modules $H_1, \ldots, H_t$ with $F_{\xi}(H) \cong H_1 \oplus \cdots \oplus H_t$ and $\mathrm{Cok}\Xi_{\Lambda^{\mathbb{L}}} F^b F(H_j) \cong G_j$, for each $j \in [1, t]$. Then, using that the functors $F^z$ reflect pregeneric modules, we know that each $H_j$ is pregeneric, and we have proved (1).

If we write $\Phi = \mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}} F^{\hat{b}} \widehat{F}$, from the diagram in the last proof, we know that $\Phi(H_{i_0}^{\mathbb{F}}) \cong G_{i_0}^{\mathbb{F}}$. Since $G$ is a direct summand of $G_{i_0}^{\mathbb{F}}$, there are morphisms $s : \Phi(H) \longrightarrow \Phi(H_{i_0}^{\mathbb{F}})$ and $p : \Phi(H_{i_0}^{\mathbb{F}}) \longrightarrow \Phi(H)$ such that $ps$ is an isomorphism. Then, there are morphisms $s' : H \longrightarrow H_{i_0}^{\mathbb{F}}$ and $p' : H_{i_0}^{\mathbb{F}} \longrightarrow H$ such that $p's'$ is an isomorphism. So $H$ is a direct summand of $H_{i_0}^{\mathbb{F}}$. □

**Theorem 4.6.** *Let $\mathcal{A}$ be an $\mathbb{L}$-constructible seminested ditalgebra over a perfect field $k$, $G$ an endofinite indecomposable $\mathcal{A}$-module, and $\mathbb{F}$ a finite field extension of $\mathbb{L}$. Then $G^{\mathbb{F}} \cong m_1 G_1 \oplus \cdots \oplus m_t G_t$, where $m_1, \ldots, m_t \in \mathbb{N}$ and $G_1, \ldots, G_t$ are pairwise non-isomorphic endofinite indecomposable $\mathcal{A}^{\mathbb{F}}$-modules. Moreover,*

1. *The $\mathcal{A}$-module $G$ is pregeneric iff each $\mathcal{A}^{\mathbb{F}}$-module $G_j$ is so.*
2. *The module $G$ is centrally finite iff each $G_j$ is so and, in this case,*

$$c\text{-endol}\,(G_j) = c\text{-endol}\,(G) \quad \textit{and} \quad c\text{-}\underline{\ell}^e\,(G_j) = c\text{-}\underline{\ell}^e\,(G) \quad \textit{for each } j \in [1, t].$$

*Proof.* If $G$ is an endofinite indecomposable $\mathcal{A}$-module, then either $G$ is finite-dimensional or $G$ is pregeneric. Then, we obtain the wanted decomposition $G^{\mathbb{F}} \cong m_1 G_1 \oplus \cdots \oplus m_t G_t$, for instance, from (4.3) and (4.4). Now, (1) is clear if we keep in mind (4.4)(1).

From [4, 4.12], (3.5), and [4, 4.9(1)], we have $\mathrm{endol}\,(G_j) = m_j \times \mathrm{endol}\,(G)$, for each $j \in [1, t]$. Then, in case $G$ is pregeneric, we get from (4.4)(2), that $G$ is centrally finite iff each $G_j$ is so, and in this case $c\text{-endol}\,(G) = c\text{-endol}\,(G_j)$ for each $j \in [1, t]$. In case $G$ is finite-dimensional, the corresponding statement follows from (4.3).

Since $\mathcal{A}$ is seminested, we have the canonical decomposition $1 = \sum_{i=1}^n e_i$ as sum of centrally primitive orthogonal idempotents of $R$ (where $(R, W)$ is the layer of $\mathcal{A}$), and then $1 = \sum_{i=1}^n e_i^{\mathbb{F}}$, where $e_i^{\mathbb{F}} = e_i \otimes 1$, is the corresponding canonical decomposition of the unit of $R^{\mathbb{F}}$ (where $(R^{\mathbb{F}}, W^{\mathbb{F}})$ is the layer of $\mathcal{A}^{\mathbb{F}}$). Then, from [4, 4.9], we obtain when $G$ is centrally finite that $c\text{-}\underline{\ell}^e\,(G_j) = c\text{-}\underline{\ell}^e\,(G)$ for each $j \in [1, t]$. □

**Definition 4.7.** Given a seminested ditalgebra $\mathcal{A}$ and a positive integer $d$, we consider the class $\mathcal{M}_{\mathcal{A}}(d)$ formed by all the finite-dimensional indecomposable $\mathcal{A}$-modules $M$ such that $c$-endol $(M) \leq d$. We will also consider the class $\mathcal{H}_{\mathcal{A}}(d)$ formed by all the centrally finite pregeneric $\mathcal{A}$-modules $H$ such that $c$-endol $(H) \leq d$.

**Corollary 4.8.** *Let $\mathcal{A}$ be an $\mathbb{L}$-constructible seminested ditalgebra over a perfect field and $\mathbb{F}$ a finite field extension of $\mathbb{L}$. Then, for any $d \in \mathbb{N}$, $\mathcal{H}_{\mathcal{A}^{\mathbb{F}}}(d)/\cong$ is finite iff $\mathcal{H}_{\mathcal{A}}(d)/\cong$ is finite.*

*Proof.* The same proof given in [15, 2.19] works here, now using centrally finite pregeneric modules, (4.5), (4.6), and (3.6). □

**Proposition 4.9.** *Let $\mathcal{A}$ be an $\mathbb{L}$-constructible seminested ditalgebra over a perfect field and $\mathbb{F}$ a finite field extension of $\mathbb{L}$. Then, $\mathcal{A}^{\mathbb{F}}$ is centrally pregenerically tame iff $\mathcal{A}$ is so. Hence, if $\mathcal{A}$ is centrally pregenerically tame, it is not almost sharply wild and it is not almost endosharply wild.*

*Proof.* The statement follows from the preceding Corollary and (3.9). □

# 5 Reduction

We shall see now how the centrally finite endofinite indecomposables with bounded central endolength, over constructible ditalgebras, can be parametrized, modulo finite field extensions, over a finite family of rational algebras. We adapt the original strategy of Drozd (see also [10] and [5]) enriched with some ideas of [4].

**Definition 5.1.** Assume that $\mathcal{A}$ is a seminested ditalgebra with layer $(R, W)$ and let $1 = \sum_{i=1}^{n} e_i$ be the decomposition of the unit of $R$ as a sum of centrally orthogonal primitive idempotents. Denote by $\mathbb{B}_0$ a fixed basis of the freely generated $R$-$R$-bimodule $W_0$ and by $\mathcal{P}_{\rm mk}$ the set of marked points of $\mathcal{A}$, respectively, see [5, 23.9]. If $M \in \mathcal{A}$-Mod is endofinite, the *endonorm* of $M$ is the number

$$||M||^e = \sum_{\alpha \in \mathbb{B}_0} \ell^e_{t(\alpha)}(M)\ell^e_{s(\alpha)}(M) + \sum_{i \in \mathcal{P}_{\rm mk}} \ell^e_i(M)^2.$$

Consequently, for $\underline{\ell}^e = (\ell^e_1, \ldots, \ell^e_n) \in \mathbb{Z}^n$, with non-negative entries, its endonorm is defined by $||\underline{\ell}^e||^e = \sum_{\alpha \in \mathbb{B}_0} \ell^e_{t(\alpha)}\ell^e_{s(\alpha)} + \sum_{i \in \mathcal{P}_{\rm mk}} (\ell^e_i)^2$.

If $M \in \mathcal{A}$-Mod is endofinite and centrally finite, then its *central endonorm* is defined by

$$||M||^c := c^2_M \times ||M||^e.$$

Notice that the endonorm introduced here for seminested ditalgebras is different from the endonorm used in [4] for admissible ditalgebras. But if $\mathcal{A}$ is admissible and seminested, it is elementary and both endonorms coincide.

**Proposition 5.2.** *Assume that $\mathcal{A}$ is an $\mathbb{L}$-constructible seminested ditalgebra over a perfect field $k$. Then, for any endofinite indecomposable $\mathcal{A}$-module $G$ and* *any finite field extension $\mathbb{F}$ of $\mathbb{L}$, in the decomposition $G^{\mathbb{F}} \cong G_1 \oplus \cdots \oplus G_t$ of the $\mathcal{A}^{\mathbb{F}}$-module $G^{\mathbb{F}}$ as a direct sum of indecomposables we have that: $G$ is finite-dimensional (resp. pregeneric, centrally finite) iff so are $G_1, \ldots, G_t$. Moreover, in the centrally finite case, we have $||G||^c = ||G_i||^c$, for all $i \in [1,t]$.*

*Proof.* This follows from (4.6). □

**Proposition 5.3.** *Let $\mathcal{A}$ be a seminested ditalgebra with layer $(R, W)$, over a perfect field $k$. Let $F^z : \mathcal{A}^z\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ be the functor associated to the reduction $\mathcal{A} \mapsto \mathcal{A}^z$ of one of the types: replacement of a layered ditalgebra by an isomorphic one, absorption of a loop, deletion of idempotents, regularization of a solid arrow, edge reduction, or unravelling. For $N \in \mathcal{A}^z\text{-Mod}$, assume that $F^z(N)$ has finite endolength and is centrally finite, then we have:*

1. *$||N||^c = ||F_\xi(N)||^c$ in case $\xi : \mathcal{A} \longrightarrow \mathcal{A}^z$ is an isomorphism of layered ditalgebras;*
2. *$||N||^c = ||F^a(N)||^c$ in the absorption case;*
3. *$||N||^c = ||F^d(N)||^c$ in the deletion of idempotents case;*
4. *$||N||^c \leq ||F^r(N)||^c$ in the regularization case, where the inequality is strict whenever $F^r(N)$ is sincere;*
5. *$||N||^c \leq ||F^e(N)||^c$ in the case of edge reduction, where the inequality is strict whenever $F^e(N)$ is sincere.*
6. *$||N||^c \leq ||F^u(N)||^c$ in the case of unravelling at a point $i_0$ using $\lambda_1, \ldots, \lambda_q$, where the inequality is strict whenever $g(x) = (x - \lambda_1) \cdots (x - \lambda_q)$ does not act invertibly on $F^u(N)$.*

*Proof.* Make $M := F^z(N)$, and recall that $F_z$ is full and faithful, thus we have an isomorphism $E_N \longrightarrow E_M$ induced by $F^z$. Make $E := E_M$. Then: if $z = a$, our claim is trivial, see [5, 25.2]; if $z = r$, our claim follows from the argument in the proof of [5, 25.3]; if $z = d$, our claim is clear, see [5, 25.4]; if $z = e$, our claim follows by the argument in the proof of [5, 25.8]; if $z = u$, our claim follows by the argument in the proof of [5, 25.9]; our first item is clear. □

**Definition 5.4.** Let $\mathcal{A}$ be a seminested ditalgebra. For any positive number $d \in \mathbb{N}$ and any non-negative integer $t$, we consider the following:

1. The symbol $\mathcal{M}_{\mathcal{A}}(d,t)$ will denote the subclass of $\mathcal{M}_{\mathcal{A}}(d)$ formed by the modules $M \in \mathcal{M}_{\mathcal{A}}(d)$ with $||M||^c \leq t$. We denote by $\mathcal{M}^0_{\mathcal{A}}(d)$ (resp. $\mathcal{M}^0_{\mathcal{A}}(d,t)$) the subclass of $\mathcal{M}_{\mathcal{A}}(d)$ (resp. $\mathcal{M}_{\mathcal{A}}(d,t)$) formed by the sincere modules in $\mathcal{M}_{\mathcal{A}}(d)$ (resp. sincere modules in $\mathcal{M}_{\mathcal{A}}(d,t)$).
2. The symbol $\mathcal{H}_{\mathcal{A}}(d,t)$ will denote the subclass of $\mathcal{H}_{\mathcal{A}}(d)$ formed by the modules $H \in \mathcal{H}_{\mathcal{A}}(d)$ with $||H||^c \leq t$. We denote by $\mathcal{H}^0_{\mathcal{A}}(d)$ (resp. $\mathcal{H}^0(d,t)$) the subclass of $\mathcal{H}_{\mathcal{A}}(d)$ (resp. $\mathcal{H}_{\mathcal{A}}(d,t)$) formed by the sincere modules in $\mathcal{H}_{\mathcal{A}}(d)$ (resp. sincere modules in $\mathcal{H}_{\mathcal{A}}(d,t)$).

We shall say that a seminested ditalgebra $\mathcal{A}$ is $(d,t)$*-trivial* (resp. *sincerely* $(d,t)$*-trivial*) iff there is only a finite number of isoclasses of modules in $\mathcal{H}_{\mathcal{A}}(d,t)$ (resp. in $\mathcal{M}^0_{\mathcal{A}}(d,t)$) and $\mathcal{H}_{\mathcal{A}}(d,t)$ (resp. $\mathcal{H}^0_{\mathcal{A}}(d,t)$) is empty.

**Remark 5.5.** Given a seminested ditalgebra $\mathcal{A}$ and $d \in \mathbb{N}$, there is a positive integer $t$ such that $\mathcal{M}_{\mathcal{A}}(d) = \mathcal{M}_{\mathcal{A}}(d,t)$, $\mathcal{M}^0_{\mathcal{A}}(d) = \mathcal{M}^0_{\mathcal{A}}(d,t)$, $\mathcal{H}_{\mathcal{A}}(d) = \mathcal{H}_{\mathcal{A}}(d,t)$, and $\mathcal{H}^0_{\mathcal{A}}(d) = \mathcal{H}^0_{\mathcal{A}}(d,t)$. Indeed, there is only a finite number of endolength vectors $\underline{\ell}^e$ and natural numbers $c$ with $c \times \sum_i \ell^e_i \leq d$, thus there are only a finite number of possibilities for $\underline{\ell}^e(M)$ and $c_M$ for any $M \in \mathcal{M}_{\mathcal{A}}(d) \bigcup \mathcal{H}_{\mathcal{A}}(d)$. Then, there is only a finite number of possibilities for the number $c_M^2 \times ||M||^e$, when $M$ runs in $\mathcal{M}_{\mathcal{A}}(d) \bigcup \mathcal{H}_{\mathcal{A}}(d)$. We can choose as $t$ any integer upper bound of these numbers.

**Proposition 5.6.** *Let $\mathcal{A}$ be an $\mathbb{L}$-constructible seminested ditalgebra over a perfect field $k$. Take $d \in \mathbb{N}$ and $t \geq 0$. Then, for any finite field extension $\mathbb{F}$ of $\mathbb{L}$, we have:*

1. *$\mathcal{M}_{\mathcal{A}^{\mathbb{F}}}(d,t)/\cong$ is finite iff $\mathcal{M}_{\mathcal{A}}(d,t)/\cong$ is finite. Similarly, $\mathcal{M}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)/\cong$ is finite iff $\mathcal{M}^0_{\mathcal{A}}(d,t)/\cong$ is finite.*
2. *$\mathcal{H}_{\mathcal{A}}(d,t) \neq \emptyset$ iff $\mathcal{H}_{\mathcal{A}^{\mathbb{F}}}(d,t) \neq \emptyset$, and $\mathcal{H}^0_{\mathcal{A}}(d,t) \neq \emptyset$ iff $\mathcal{H}^0_{\mathcal{A}^{\mathbb{F}}}(d,t) \neq \emptyset$.*
3. *The $\mathbb{F}$-ditalgebra $\mathcal{A}^{\mathbb{F}}$ is $(d,t)$-trivial (resp. sincerely $(d,t)$-trivial) iff the $\mathbb{L}$-ditalgebra $\mathcal{A}$ is $(d,t)$-trivial (resp. sincerely $(d,t)$-trivial).*

*Proof.* Consider the scalar restriction functor $F_\xi : \mathcal{A}^{\mathbb{F}}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$.

(1) We show first that any (resp. sincere) $N \in \mathcal{M}_{\mathcal{A}^{\mathbb{F}}}(d,t)$ is a direct summand of $M^{\mathbb{F}}$ for some (resp. sincere) $M \in \mathcal{M}_{\mathcal{A}}(d,t)$. Given $N \in \mathcal{M}_{\mathcal{A}^{\mathbb{F}}}(d,t)$, we have a decomposition $F_\xi(N) \cong M_1 \oplus \cdots \oplus M_u$ as a direct sum of indecomposables in $\mathcal{A}$-mod, and we have decompositions $M_i^{\mathbb{F}} \cong N_{i,1} \oplus \cdots \oplus N_{i,v_i}$ as direct sum of indecomposables in $\mathcal{A}^{\mathbb{F}}$-mod. By [3, 3.7], we know that $N$ is a direct summand of $F_\xi(N)^{\mathbb{F}} \cong M_1^{\mathbb{F}} \oplus \cdots \oplus M_u^{\mathbb{F}}$. Thus, $N \cong N_{i,j}$ for some $i,j$. Then, from (4.6) and (5.2), we obtain $c\text{-endol}\,(M_i) = c\text{-endol}\,(N_{i,j}) = c\text{-endol}\,(N) \leq d$ and $||M_i||^c = ||N_{i,j}||^c = ||N||^c \leq t$. Thus, $M_i \in \mathcal{M}_{\mathcal{A}}(d,t)$. If $N$ is sincere, so is $N_{i,j}$ and $M_i^{\mathbb{F}}$, thus $M_i$ is also sincere. Then, $\mathcal{M}_{\mathcal{A}}(d,t)/\cong$ finite implies that $\mathcal{M}_{\mathcal{A}^{\mathbb{F}}}(d,t)/\cong$ is finite, and similarly for the sincere case.

Given $M \in \mathcal{M}_{\mathcal{A}}(d,t)$, from (4.6) and (5.2), we know that in the decomposition $M^{\mathbb{F}} \cong M_1 \oplus \cdots \oplus M_u$ as a direct sum of idecomposable $\mathcal{A}^{\mathbb{F}}$-modules, we have $M_i \in \mathcal{M}_{\mathcal{A}^{\mathbb{F}}}(d,t)$. Moreover, if $M$ is sincere and $\mathbb{B}$ is a basis for the $\mathbb{L}$-vector space $\mathbb{F}$, then $\oplus_{\mathbb{B}} M \cong F_\xi(M^{\mathbb{F}}) \cong \oplus_j F_\xi(M_j)$ and $F_\xi(M_j)$ is sincere for all $j$, thus $M_j$ is sincere for all $j$. If $\mathcal{M}_{\mathcal{A}^{\mathbb{F}}}(d,t)/\cong$ is finite, there are only finitely many isomorphism classes of such modules $M_i$, then from (3.6), we know that there are only finitely many possible isoclasses of such modules $M$. Then, $\mathcal{M}_{\mathcal{A}^{\mathbb{F}}}(d,t)/\cong$ finite implies that $\mathcal{M}_{\mathcal{A}}(d,t)/\cong$ is finite, and similarly for the sincere case.

(2) Given $G \in \mathcal{H}_{\mathcal{A}}(d,t)$, from (4.6) and (5.2), we have a decomposition $G^{\mathbb{F}} \cong G_1 \oplus \cdots \oplus G_u$ as a direct sum of pregeneric $\mathcal{A}^{\mathbb{F}}$-modules $G_1, \ldots, G_u \in \mathcal{H}_{\mathcal{A}^{\mathbb{F}}}(d,t)$. As in the proof of (1), we can show that if $G$ is sincere then each $G_i$ is sincere.

Take $G \in \mathcal{H}_{\mathcal{A}^{\mathbb{F}}}(d,t)$. From (4.5), we get $F_\xi(G) \cong G_1 \oplus \cdots \oplus G_t$, for some pregeneric $\mathcal{A}$-modules $G_1, \ldots, G_t$ and $G$ is a direct summand of $G_i^{\mathbb{F}}$, for some $i \in [1,t]$. Using (4.6), we have a direct sum decomposition $G_i^{\mathbb{F}} \cong G \oplus H_2 \oplus \cdots \oplus H_s$, with $H_i$ pregeneric. Moreover, $G$ centrally finite implies that $G_i$ is centrally finite with $c$-endol$(G_i) = c$-endol$(G) \leq d$ and $||G_i||^c = ||G||^c \leq t$. Thus, $G_i \in \mathcal{H}_{\mathcal{A}}(d,t)$. Finally, notice that if $G$ is sincere, so is $G_i^{\mathbb{F}}$, and so is $G_i$.

(3) This clearly follows from *1* and *2*. ☐

**Remark 5.7.** If $\mathcal{A}$ is an $\mathbb{L}$-constructible seminested ditalgebra and $d \in \mathbb{N}$, the same number $t$ chosen in (5.5) satisfies that $\mathcal{M}_{\mathcal{A}^{\mathbb{F}}}(d) = \mathcal{M}_{\mathcal{A}^{\mathbb{F}}}(d,t)$, $\mathcal{M}^0_{\mathcal{A}^{\mathbb{F}}}(d) = \mathcal{M}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$, $\mathcal{H}_{\mathcal{A}^{\mathbb{F}}}(d) = \mathcal{H}_{\mathcal{A}^{\mathbb{F}}}(d,t)$, and $\mathcal{H}^0_{\mathcal{A}^{\mathbb{F}}}(d) = \mathcal{H}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$, for any finite field extension $\mathbb{F}$ of $\mathbb{L}$.

**Theorem 5.8.** *Let $\mathcal{A}$ be an $\mathbb{L}$-constructible seminested ditalgebra over a perfect field $k$. Suppose that $\mathcal{A}$ is not almost sharply wild or not almost endosharply wild. Then, for any non-negative integer $d$ and $t \geq 0$, there is a finite field extension $\mathbb{F}_\omega$ of $\mathbb{L}$ such that: for any finite field extension $\mathbb{F}$ of $\mathbb{F}_\omega$, there are minimal $\mathbb{F}$-ditalgebras $\mathcal{B}_1, \ldots, \mathcal{B}_p$, and functors $F_1, \ldots, F_p$ such that:*

1. *Each functor $F_i : \mathcal{B}_i\text{-Mod} \longrightarrow \mathcal{A}^{\mathbb{F}}\text{-Mod}$ is full and faithful and preserves endofinite modules;*

2. *For almost every $M \in \mathcal{M}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$ there are $i \in [1,p]$ and $N \in \mathcal{B}_i\text{-Mod}$ with $F_i(N) \cong M$ in $\mathcal{A}^{\mathbb{F}}\text{-Mod}$;*

3. *For every $G \in \mathcal{H}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$ there are $i \in [1,p]$ and a principal generic $B_i$-module $Q_j$, with $F_i(Q_j) \cong G$ in $\mathcal{A}^{\mathbb{F}}\text{-Mod}$, see [5, 31.3].*

4. *The functors $F_i$ are compositions of basic reduction functors: thus each $\mathcal{B}_i$ is obtained from $\mathcal{A}^{\mathbb{F}}$ by a finite sequence of basic operations of the form $\mathcal{C} \mapsto \mathcal{C}^z$, where $z \in \{a,r,d,e,u\}$, or there is an isomorphism of layered ditalgebras $\xi : \mathcal{C} \longrightarrow \mathcal{C}^z$, and $F_i$ is a composition of the corresponding basic reduction functors $F^z : \mathcal{C}^z\text{-Mod} \longrightarrow \mathcal{C}\text{-Mod}$, or $F^z = F_\xi : \mathcal{C}^z\text{-Mod} \longrightarrow \mathcal{C}\text{-Mod}$ given by restriction through the isomorphism $\xi$.*

*Proof.* This proof is an adaptation of the proof of Drozd's Tame and Wild Theorem, more precisely of [4, 7.4] and [5, 26.9]. We give a full proof for the not almost sharply wild case, the not almost endosharply wild case is similar.

Suppose that $\mathcal{A}$ is an $\mathbb{L}$-constructible seminested ditalgebra not almost sharply wild. Then, for each finite field extension $\mathbb{F}$ of $\mathbb{L}$, from (3.4), we know that $\mathcal{A}^{\mathbb{F}}$ is an $\mathbb{F}$-constructible seminested ditalgebra not almost sharply wild. The same will remain true for any ditalgebra obtained from $\mathcal{A}^{\mathbb{F}}$ by a finite number of basic operations of type $\mathcal{A}^{\mathbb{F}} \mapsto \mathcal{A}^{\mathbb{F}z}$ with $z \in \{a,r,d,e,u\}$, see (3.11).

We shall proceed by induction on $t$, for every $d \in \mathbb{N}$.

If $\mathcal{A}$ is sincerely $(d,0)$-trivial, by (5.6), so is $\mathcal{A}^{\mathbb{F}}$ for any finite field extension $\mathbb{F}$ of $\mathbb{L}$, and we have nothing to show (the empty family of functors works for any $\mathcal{A}^{\mathbb{F}}$). So assume that $t > 0$ and that, for any $t' < t$, any $d' \in \mathbb{N}$, and any $\mathbb{L}'$-constructible seminested ditalgebra $\mathcal{A}'$, which is not almost sharply wild, there is a finite field extension $\mathbb{F}'_\omega$ of $\mathbb{L}'$ and, for each finite field extension $\mathbb{F}$ of $\mathbb{F}'_\omega$, there are functors $F_i : \mathcal{B}_i\text{-Mod}\longrightarrow\mathcal{A}'^{\mathbb{F}}\text{-Mod}$, with $\mathcal{B}_i$ a minimal $\mathbb{F}$-ditalgebra, satisfying *1–4*, for $\mathcal{A}'$, $d'$ and $t'$. In particular, for almost every $M \in \mathcal{M}^0_{\mathcal{A}'^{\mathbb{F}}}(d', t')$ there are $i$ and $N \in \mathcal{B}_i$-mod such that $F_i(N) \cong M$ in $\mathcal{A}'^{\mathbb{F}}$-Mod, and for each $G \in \mathcal{H}^0_{\mathcal{A}'^{\mathbb{F}}}(d', t')$ there are a unique $i$ and a principal generic $B_i$-module $Q_j$ with $F_i(Q_j) \cong G$.

Now, fix any $d \in \mathbb{N}$ and assume that the $\mathbb{L}$-constructible seminested ditalgebra $\mathcal{A}$ is not sincerely $(d,t)$-trivial. Otherwise, from (5.6), there is nothing to show: the empty family of functors works for any $\mathcal{A}^{\mathbb{F}}$, and any finite field extension $\mathbb{F}$ of $\mathbb{L}$.

Since $\mathcal{A}$ is a seminested ditalgebra, we can choose a minimal solid arrow $\alpha : i_0\longrightarrow j_0$ in $\mathbb{B}_0$, that is a solid arrow $\alpha$ with minimal height. Then, by triangularity, we have that $\delta(\alpha) \in W_1$, where $(R,W)$ denotes the seminested layer of $\mathcal{A}$, see [5, 23.5].

- *Case 1:* $\delta(\alpha) = 0$ and $i_0 = j_0$.

Since $\mathcal{A}$ is not almost sharply wild, by (3.14), $Re_{i_0} \cong \mathbb{L}$. Consider the ditalgebra $\mathcal{A}^a$ obtained from $\mathcal{A}$ by absorption of the loop $\alpha$. By [5, 20.6], we can identify $\mathcal{A}^{a\mathbb{F}}$ with $\mathcal{A}^{\mathbb{F}a}$, for each finite field extension $\mathbb{F}$ of $\mathbb{L}$, and we have the associated functor $\widehat{F}^a : \mathcal{A}^{\mathbb{F}a}\text{-Mod}\longrightarrow\mathcal{A}^{\mathbb{F}}\text{-Mod}$. Then, we can apply (5.3)(2) to every $M \in \mathcal{M}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$ (resp. $M \in \mathcal{H}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$) to obtain $N \in \mathcal{M}^0_{\mathcal{A}^{\mathbb{F}a}}(d,t)$ (resp. $N \in \mathcal{H}^0_{\mathcal{A}^{\mathbb{F}a}}(d,t)$) with $\widehat{F}_a(N) \cong M$. We have that $\mathcal{A}^a$ has one solid arrow less that $\mathcal{A}$. Repeating this argument, if necessary, either we end up with a seminested ditalgebra with no solid arrows (that is a minimal ditalgebra) or at some step we obtain a ditalgebra with a minimal solid arrow $\alpha$ in one of the following cases.

- *Case 2:* $\delta(\alpha) = 0$ and $i_0 \neq j_0$.

Since $\mathcal{A}$ is not almost sharply wild, by (3.14), $Re_{i_0} \cong \mathbb{L}$ and $Re_{j_0} \cong \mathbb{L}$. Then, we can consider the ditalgebra $\mathcal{A}^e$ obtained from $\mathcal{A}$ by reduction of the edge $\alpha$ and the equivalence functor $F^e : \mathcal{A}^e\text{-Mod}\longrightarrow\mathcal{A}\text{-Mod}$. Then, from (5.3)(5) and (2.9), for every module $M \in \mathcal{M}^0_{\mathcal{A}}(d,t)$ (resp. $M \in \mathcal{H}^0_{\mathcal{A}}(d,t)$) we obtain a module $N \in \mathcal{M}^0_{\mathcal{A}^e}(d,t-1)$ (resp. $N \in \mathcal{H}^0_{\mathcal{A}^e}(d,t-1)$), with $F_e(N) \cong M$. By assumption $\mathcal{A}$ is not sincerely $(d,t)$-trivial, hence $\mathcal{A}^e$ is not sincerely $(d,t-1)$-trivial. Applying our induction hypothesis to $\mathcal{A}^e$, $d$, and $t-1$, we get a finite field extension $\mathbb{F}_\omega$ of $\mathbb{L}$ such that for any finite field extension $\mathbb{F}$ of $\mathbb{F}_\omega$, we have functors $F_i : \mathcal{B}_i\text{-Mod}\longrightarrow\mathcal{A}^{e\mathbb{F}}\text{-Mod}$, $i \in [1,m]$, satisfying the corresponding statements *1–4*. Recall from [5, 20.11] that the seminested ditalgebras $\mathcal{A}^{e\mathbb{F}}$ and $\mathcal{A}^{\mathbb{F}e}$ can be identified, and we have an edge reduction equivalence functor $\hat{F}^e : \mathcal{A}^{e\mathbb{F}}\text{-Mod} = \mathcal{A}^{\mathbb{F}e}\text{-Mod}\longrightarrow\mathcal{A}^{\mathbb{F}}\text{-Mod}$. As before, for every module $M \in \mathcal{M}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$ (resp. $M \in \mathcal{H}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$) we obtain a module $N \in \mathcal{M}^0_{\mathcal{A}^{\mathbb{F}e}}(d,t-1)$ (resp. $N \in \mathcal{H}^0_{\mathcal{A}^{\mathbb{F}e}}(d,t-1)$), with $\widehat{F}_e(N) \cong M$. Then, $\mathcal{F}_{\mathbb{F}} := \{\hat{F}^e F_i \mid i \in [1,m]\}$ is the required family of functors for $\mathcal{A}^{\mathbb{F}}$, $d$, and $t$.

- *Case 3:* $\delta(\alpha) \neq 0$.

Write $C := Re_{i_0} \otimes_{\mathbb{L}} Re_{j_0}$. Then, $e_{j_0}W_1e_{i_0}$ is a $C$-module and we can write $\delta(\alpha) = \sum_{i=1}^{j} c_iv_i$, for some $v_1, \ldots, v_j \in e_{j_0}\mathbb{B}_1e_{i_0}$ and $0 \neq c_i \in C$.

- *Subcase 3.1:* Some $c_t$ is invertible in $C$.

This is the case, for instance, if $Re_{i_0} \cong \mathbb{L}$ and $Re_{j_0} \cong \mathbb{L}$. Make $v'_t := \sum_{i=1}^{j} c_iv_i$ and consider the change of basis for $W_1$ where $v_1, \ldots, v_t, \ldots, v_j$ is replaced by $v_1, \ldots, v'_t, \ldots, v_j$ using [5, 26.1] (more precisely, consider the matrix $Q$ with $Q_{ti} = c_i$ for $i \in [1, j]$; $Q_{ii} = 1$ for $i \neq t$; and $Q_{iq} = 0$ for $t \neq i \neq q$; where $Q$ is invertible because $c_t$ is so).

Then, we can apply regularization to $\mathcal{A}$ and obtain an associated equivalence functor $F_r : \mathcal{A}^r\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$. Then, we proceed as in *Case 2*. Namely, from (5.3)(4) and (2.8), for every module $M \in \mathcal{M}^0_{\mathcal{A}}(d, t)$ (resp. $M \in \mathcal{H}^0_{\mathcal{A}}(d, t)$) we obtain a module $N \in \mathcal{M}^0_{\mathcal{A}^r}(d, t-1)$ (resp. $N \in \mathcal{H}^0_{\mathcal{A}^r}(d, t-1)$), with $F_r(N) \cong M$. By assumption $\mathcal{A}$ is not sincerely $(d, t-1)$-trivial, hence $\mathcal{A}^r$ is not sincerely $(d, t-1)$-trivial. Applying our induction hypothesis to $\mathcal{A}^r$, $d$, and $t-1$, we get a finite field extension $\mathbb{F}_\omega$ of $\mathbb{L}$ such that for any finite field extension $\mathbb{F}$ of $\mathbb{F}_\omega$, we have the corresponding family of functors $F_i : \mathcal{B}_i\text{-Mod} \longrightarrow \mathcal{A}^{r\mathbb{F}}\text{-Mod}$, $i \in [1, m]$. Recall from [5, 20.5] that the seminested ditalgebras $\mathcal{A}^{r\mathbb{F}}$ and $\mathcal{A}^{\mathbb{F}r}$ can be identified, and we have a regularization equivalence functor $\hat{F}^r : \mathcal{A}^{r\mathbb{F}}\text{-Mod} = \mathcal{A}^{\mathbb{F}r}\text{-Mod} \longrightarrow \mathcal{A}^{\mathbb{F}}\text{-Mod}$. Again, for every module $M \in \mathcal{M}^0_{\mathcal{A}^{\mathbb{F}}}(d, t)$ (resp. $M \in \mathcal{H}^0_{\mathcal{A}^{\mathbb{F}}}(d, t)$) we obtain a module $N \in \mathcal{M}^0_{\mathcal{A}^{\mathbb{F}r}}(d, t-1)$ (resp. $N \in \mathcal{H}^0_{\mathcal{A}^{\mathbb{F}r}}(d, t-1)$), with $\widehat{F}_r(N) \cong M$. Then, $\mathcal{F}_{\mathbb{F}} := \{\hat{F}^rF_i \mid i \in [1, m]\}$ is the required family of functors for $\mathcal{A}^{\mathbb{F}}$, $d$, and $t$.

- *Subcase 3.2:* $Re_{j_0} \cong \mathbb{L}$.

Since we have already considered the *Subcase 3.1*, we may assume that $Re_{i_0} \not\cong \mathbb{L}$. Thus, $Re_{i_0} = \mathbb{L}[x]_{f(x)}$ and $C \cong \mathbb{L}[x]_{f(x)}$. By an appropiate change of basis of $W_1$ of the form $v'_i = f(x)^{-p}v_i$, for all $i$, using again [5, 26.1], we may assume that $c_i \in \mathbb{L}[x]$. Performing a finite field extension of $\mathbb{L}$ if necessary, we can assume that the polynomial $c_1(x)$ splits in $\mathbb{L}[x]$ as a product of linear factors. Consider the ditalgebra $\mathcal{A}^u$ obtained from $\mathcal{A}$ by unravelling at $i_0$ using $d$ and the different roots of $c_1(x)$, see [5, 23.23].

From (5.3)(6) and (2.10), for every module $M \in \mathcal{M}^0_{\mathcal{A}}(d, t)$ (resp. $M \in \mathcal{H}^0_{\mathcal{A}}(d, t)$) with $M(c_1(x))$ not invertible, there is a module $N \in \mathcal{M}^0_{\mathcal{A}^u}(d, t-1)$ (resp. $N \in \mathcal{H}^0_{\mathcal{A}^u}(d, t-1)$), with $F_u(N) \cong M$. Applying our induction hypothesis to $\mathcal{A}^u$ we get a finite field extension $\mathbb{F}_\omega$ of $\mathbb{L}$ such that for any finite field extension $\mathbb{F}$ of $\mathbb{F}_\omega$, we have the corresponding family of functors $F_i : \mathcal{B}_i\text{-Mod} \longrightarrow \mathcal{A}^{u\mathbb{F}}\text{-Mod}$, $i \in [1, m]$. Recall from [5, 20.11] that the seminested ditalgebras $\mathcal{A}^{u\mathbb{F}}$ and $\mathcal{A}^{\mathbb{F}u}$ can be identified, and we have an associated unravelling functor $\hat{F}^u : \mathcal{A}^{u\mathbb{F}}\text{-Mod} = \mathcal{A}^{\mathbb{F}u}\text{-Mod} \longrightarrow \mathcal{A}^{\mathbb{F}}\text{-Mod}$.

Then, $\{\hat{F}^uF_i \mid i \in [1, m]\}$ is a family of functors as in *1* and *4*, for $\mathcal{A}^{\mathbb{F}}$, $d$, and $t$, which covers only $\mathcal{A}^{\mathbb{F}}$-modules $M$ as in *2* or *3* with $M(c_1(x))$ not invertible.

So, consider also the ditalgebra $\mathcal{A}^l$ obtained from $\mathcal{A}$ by localizing at the vertex $i_0$ using the polynomial $c_1(x)$, see [5, 26.4]. Then for every module $M \in \mathcal{M}^0_{\mathcal{A}}(d, t)$ (resp. $M \in \mathcal{H}^0_{\mathcal{A}}(d, t)$) with $M(c_1(x))$ invertible, we obtain a module $N \in \mathcal{M}^0_{\mathcal{A}^l}(d,t)$ (resp. $N \in \mathcal{H}^0_{\mathcal{A}^l}(d,t)$), with $F^l(N) \cong M$, see [5, 26.6]. From [5, 26.7], we know that $\mathcal{A}^l$ can be identified with some $\mathcal{A}^{ud}$ and $F^l$ with $F^u F^d$.

According to the description of the differential $\delta^l$ of $\mathcal{A}^l$ given in [5, 26.4 and 17.7], we have that $\delta^l(1 \otimes \alpha \otimes 1) = \sum_{i=1}^{j} c_i(x)(1 \otimes v_i \otimes 1)$. But now, $c_1$ is invertible and we can proceed as in *Subcase 3.1* to produce a finite field extension $\mathbb{F}'_\omega$ of $\mathbb{L}$ such that for any field extension $\mathbb{F}$ of $\mathbb{F}'_\omega$ there is a family $F'_j : \mathcal{B}'_j\text{-Mod} \longrightarrow \mathcal{A}^{l\mathbb{F}}\text{-Mod}$, $j \in [1,m']$, which satisfies *1-4* for $\mathcal{A}^{l\mathbb{F}}$, $d$, and $t$.

Now, consider a finite field extension $\mathbb{F}''_\omega$ of $\mathbb{L}$ containing $\mathbb{F}_\omega$ and $\mathbb{F}'_\omega$, and a finite field extension $\mathbb{F}$ of $\mathbb{F}''_\omega$.

From our preceding remark on $\mathcal{A}^l$, [5, 30.4 and 20.11], we can identify $\mathcal{A}^{l\mathbb{F}}$ with $\mathcal{A}^{\mathbb{F}l}$, and we have an associated functor $\hat{F}^l : \mathcal{A}^{l\mathbb{F}}\text{-Mod} = \mathcal{A}^{\mathbb{F}l}\text{-Mod} \longrightarrow \mathcal{A}^{\mathbb{F}}\text{-Mod}$ such that for any $M \in \mathcal{M}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$ (resp. $M \in \mathcal{H}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$) with $M(c_1(x))$ invertible, we obtain a module $N \in \mathcal{M}^0_{\mathcal{A}^{\mathbb{F}l}}(d,t)$ (resp. $N \in \mathcal{H}^0_{\mathcal{A}^{\mathbb{F}l}}(d,t)$), with $\widehat{F}^l(N) \cong M$.

Then, $\mathcal{F}_{\mathbb{F}} := \{\hat{F}^u F_i \mid i \in [1,m]\} \bigcup \{\hat{F}^l F'_j \mid j \in [1,m']\}$ is a family of functors as in *1* and *4*, for $\mathcal{A}^{\mathbb{F}}$, $d$, and $t$, which covers the required $\mathcal{A}^{\mathbb{F}}$-modules in *2-3*.

- *Subcase 3.3:* $Re_{i_0} \cong \mathbb{L}$.

This case is dual to *Subcase 3.2*.

- *Subcase 3.4:* $Re_{i_0} \ncong \mathbb{L}$ and $Re_{j_0} \ncong \mathbb{L}$.

Assume that $Re_{i_0} = \mathbb{L}[x]_{f(x)}$ and $Re_{j_0} = \mathbb{L}[y]_{g(y)}$. Hence, $C = \mathbb{L}[x,y]_{f(x)g(y)}$. After an appropiate change of basis of $W_1$, of the form $v'_i = f(x)^{-p} g(y)^{-q} v_i$, using [5, 26.1], we may assume that all the $c_i$ are polynomials in $\mathbb{L}[x,y]$. Let $h(x,y)$ be the highest common factor of the $c_i(x,y)$ and assume that $h(x,y)q_i(x,y) = c_i(x,y)$, for all $i$. Since the $q_i(x,y)$ are coprime in $\mathbb{L}(x)[y]$, there are polynomials $s_i(x,y) \in \mathbb{L}[x,y]$ and a non-zero polynomial $c(x) \in \mathbb{L}[x]$ such that

$$c(x) = \sum_{i=1}^{j} s_i(x,y) q_i(x,y).$$

Again, performing a finite field extension of $\mathbb{L}$ if necessary, we can assume that $c(x)$ splits in $\mathbb{L}[x]$ as a product of linear factors. Then, we can proceed as in *Subcase 3.2*, by unravelling at $i_0$ using $d$ and the different roots of $c(x)$. Moreover, we can consider the ditalgebra $\mathcal{A}^l$ obtained from $\mathcal{A}$ by localizing at the point $i_0$ using $c(x)$. We have the associated functors $\widehat{F}^u : \mathcal{A}^{\mathbb{F}u}\text{-Mod} \longrightarrow \mathcal{A}^{\mathbb{F}}\text{-Mod}$ and $\widehat{F}^l : \mathcal{A}^{\mathbb{F}l}\text{-Mod} \longrightarrow \mathcal{A}^{\mathbb{F}}\text{-Mod}$, as before.

Then, any $M \in \mathcal{M}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$ (resp. $M \in \mathcal{H}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$) with $M(c(x))$ non-invertible, can be covered as in *Subcase 3.2*, for any finite field extension $\mathbb{F}$ of a suitable finite field extension $\mathbb{F}_\omega$ of $\mathbb{L}$. If $M(c(x))$ is invertible, we have that $M \cong \widehat{F}^l(N)$ where $N \in \mathcal{A}^{\mathbb{F}l}$-Mod is sincere and has the same endonorm than $M$. From the preceding formula for $c(x)$, we obtain in the ditalgebra $\mathcal{A}^l$ that $1 = \sum_{i=1}^{j} [s_i(x,y)c(x)^{-1}] q_i(x,y)$. The terms in this formula all belong to the algebra $H := \mathbb{L}[x,y]_{c(x)}$. Here, $H \cong D[y]$, where $D = \mathbb{L}[x]_{c(x)}$ is a principal ideal domain. From [5, 26.2], we obtain an invertible matrix $Q \in M_{j\times j}(H)$ with first row $(q_1, \dots, q_j)$. Consider the change of basis for $W_1^l = S \otimes_R W_1 \otimes_R S$, which replaces each $w_i := 1 \otimes v_i \otimes 1$ by $w'_i$ defined by the formula

$$(w'_1, \dots, w'_j)^t = Q(w_1, \dots, w_j)^t.$$

Hence,

$$\delta^l(1 \otimes \alpha \otimes 1) = \sum_i c_i w_i = \sum_i h q_i w_i = h w'_1.$$

Since $\mathcal{A}^l$ is not almost sharply wild, by (3.14), we can assume, after performing a finite field extension if necessary, that $h$ is invertible. Now, we can replace the basis $w'_1, w'_2, \dots$ of $W_1^l$ by $hw'_1, w'_2, \dots$ and apply regularization, as in *Subcase 3.1*, to finish the proof. □

**Corollary 5.9.** *Let $\mathcal{A}$ be an $\mathbb{L}$-constructible seminested ditalgebra over a perfect field $k$. Suppose that $\mathcal{A}$ is not almost sharply wild or not almost endosharply wild. Then, for any non-negative integer $d$ and $t \geq 0$, there is a finite field extension $\mathbb{F}_\omega$ of $\mathbb{L}$ such that: for any finite field extension $\mathbb{F}$ of $\mathbb{F}_\omega$, there are rational $\mathbb{F}$-algebras $\Gamma_1, \dots, \Gamma_q$, and functors $F_1, \dots, F_q$ such that:*

1. *Each functor $F_i : \Gamma_i\text{-Mod} \longrightarrow \mathcal{A}^{\mathbb{F}}\text{-Mod}$ is sharp endosharp and $F_i(\mathbb{F}(x))$ is an algebraically rigid centrally finite pregeneric $\mathcal{A}^{\mathbb{F}}$-module.*
2. *For almost every $M \in \mathcal{M}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$ there are $i \in [1,q]$ and $N \in \Gamma_i\text{-Mod}$ with $F_i(N) \cong M$ in $\mathcal{A}^{\mathbb{F}}\text{-Mod}$;*
3. *For every $G \in \mathcal{H}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$ there is a unique $i \in [1,q]$ with $F_i(\mathbb{F}(x)) \cong G$ in $\mathcal{A}^{\mathbb{F}}\text{-Mod}$.*
4. *For each $i \in [1,q]$ there is an $A^{\mathbb{F}}$-$\Gamma_i$-bimodule $Y_i$, which is free of finite rank as a right $\Gamma_i$-module, such that $F_i \cong L_{\mathcal{A}^{\mathbb{F}}}(Y_i \otimes_{\Gamma_i} -)$, where $L_{\mathcal{A}^{\mathbb{F}}} : A^{\mathbb{F}}\text{-Mod} \longrightarrow \mathcal{A}^{\mathbb{F}}\text{-Mod}$ is the canonical embedding.*

*Proof.* Apply the last theorem to $\mathcal{A}$, $d$ and $t$, to obtain the field $\mathbb{F}_\omega$, then fix some finite field extension $\mathbb{F}$ of $\mathbb{F}_\omega$ to get minimal $\mathbb{F}$-ditalgebras $\mathcal{B}_1, \dots, \mathcal{B}_p$ and functors $F_1, \dots, F_p$ as before. Since each $\mathcal{B}_i$ is a minimal $\mathbb{F}$-ditalgebra, we know that $B_i \cong \Gamma_{i,1} \times \cdots \times \Gamma_{i,s_i} \times \mathbb{F} \times \cdots \times \mathbb{F}$, where $\Gamma_{i,j}$ is a rational $\mathbb{F}$-algebra for all $j \in [1, s_i]$. Consider the canonical injections $\Theta_{i,j} : \Gamma_{i,j}\text{-Mod} \longrightarrow B_i\text{-Mod}$ and the canonical embedding functor $L_{\mathcal{B}_i} : B_i\text{-Mod} \longrightarrow \mathcal{B}_i\text{-Mod}$. Then, we have a natural isomorphism $\Theta_{i,j} \cong \Gamma_{i,j} \otimes_{\Gamma_{i,j}} -$, where $\Gamma_{i,j}$ has the natural $B_i$-$\Gamma_{i,j}$-bimodule structure. Since $\mathcal{B}_i$ is minimal, it is also clear that $L_{\mathcal{B}_i}$ coincides with the extension functor $E_i : B_i\text{-Mod} \longrightarrow \mathcal{B}_i\text{-Mod}$ corresponding to the proper subalgebra $B_i$ of $\mathcal{B}_i$. Then, from (3.10), we know that $L_{\mathcal{B}_i}$ is a sharp endosharp functor. It is clear that $\Theta_{i,j}$ is also a sharp endosharp functor. From (5.8)(1) and (3.8)(2), $F_i$ is sharp and endosharp. From [5, 22.7] and (3.8)(1), we get a sharp endosharp functor $F_{i,j} := F_i E_i \Theta_{i,j} \cong F_i L_{\mathcal{B}_i}(\Gamma_{i,j} \otimes_{\Gamma_{i,j}} -) \cong L_{\mathcal{A}^{\mathbb{F}}}(Y_{i,j} \otimes_{\Gamma_{i,j}} -)$, where $Y_{i,j}$ is the $A^{\mathbb{F}}$-$\Gamma_{i,j}$-bimodule $F_i(\Gamma_{i,j})$, which is free of finite rank by the right.

It is clear that $\Theta_{i,j}$ preserves algebraically rigid centrally finite pregeneric modules, and so does $E_i$ by (2.6). The functor $F_i$ preserves algebraically rigid centrally finite pregeneric modules by (2.8) and (2.10). Then, $F_{i,j}$ also has these properties.

Finally, notice that almost every finite-dimensional indecomposable module $M$ of $B_i$-Mod is of the form $E_i\Theta_{i,j}(M')$, for some $j$ and some $M' \in \Gamma_{i,j}$-Mod, and every principal pregeneric $B_i$-module $Q$ is of the form $E_i\Theta_{i,j}(\mathbb{F}(x))$, for a unique $j$. Then, the family of rational $\mathbb{F}$-algebras $\{\Gamma_{i,j}\}_{i,j}$ and the family of functors $\{F_{i,j}\}_{i,j}$ work. □

In the following result we remove the sincerity and the endonorm from the statement of the last corollary.

**Theorem 5.10.** *Let $\mathcal{A}$ be an $\mathbb{L}$-constructible seminested ditalgebra over a perfect field $k$. Assume that $\mathcal{A}$ is not almost sharply wild or not almost endosharply wild. Let $d$ be a non-negative integer. Then, there is a finite field extension $\mathbb{F}_\omega$ of $\mathbb{L}$ such that: for any finite field extension $\mathbb{F}$ of $\mathbb{F}_\omega$, there are rational $\mathbb{F}$-algebras $\Gamma_1, \ldots, \Gamma_m$ and functors $F_1, \ldots, F_m$ such that:*

1. *Each functor $F_i : \Gamma_i\text{-Mod} \longrightarrow \mathcal{A}^{\mathbb{F}}\text{-Mod}$ is sharp endosharp and $F_i(\mathbb{F}(x))$ is an algebraically rigid centrally finite pregeneric $\mathcal{A}^{\mathbb{F}}$-module.*
2. *For almost every $M \in \mathcal{M}_{\mathcal{A}^{\mathbb{F}}}(d)$ there are $i \in [1,m]$ and $N \in \Gamma_i\text{-Mod}$ with $F_i(N) \cong M$ in $\mathcal{A}^{\mathbb{F}}\text{-Mod}$;*
3. *For every $G \in \mathcal{H}_{\mathcal{A}^{\mathbb{F}}}(d)$ there is a unique $i \in [1,m]$ with $F_i(\mathbb{F}(x)) \cong G$ in $\mathcal{A}^{\mathbb{F}}\text{-Mod}$.*
4. *For each $i \in [1,m]$ there is an $A^{\mathbb{F}}$-$\Gamma_i$-bimodule $Y_i$, which is free of finite rank as a right $\Gamma_i$-module, such that $F_i \cong L_{\mathcal{A}^{\mathbb{F}}}(Y_i \otimes_{\Gamma_i} -)$, where $L_{\mathcal{A}^{\mathbb{F}}} : A^{\mathbb{F}}\text{-Mod} \longrightarrow \mathcal{A}^{\mathbb{F}}\text{-Mod}$ is the canonical embedding.*

*Proof.* This is a standard procedure and similar to the proof of [4, 8.1]. Anyway, we recall the argument. First notice that we have a corollary of (5.9) by eliminating the numbers $t$ in the hypothesis of (5.9) and replacing $\mathcal{M}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$ (resp. $\mathcal{H}^0_{\mathcal{A}^{\mathbb{F}}}(d,t)$) by $\mathcal{M}^0_{\mathcal{A}^{\mathbb{F}}}(d)$ (resp. by $\mathcal{H}^0_{\mathcal{A}^{\mathbb{F}}}(d)$) in its conclusions. The proof of this corollary, which we will refer to as Corollary (5.9)', is trivial if we keep in mind (5.5) and (5.7).

Now take an $\mathbb{L}$-constructible seminested ditalgebra $\mathcal{A}$ and assume that it is not almost sharply wild. The not almost endosharply wild case is treated in a similar way.

Given $d \in \mathbb{N}$, we shall say that a seminested ditalgebra $\mathcal{A}$ is *$d$-trivial* (resp. *sincerely $d$-trivial*) iff there is only a finite number of isoclasses of modules in $\mathcal{M}_{\mathcal{A}}(d)$ (resp. in $\mathcal{M}^0_{\mathcal{A}}(d)$), and $\mathcal{H}_{\mathcal{A}}(d)$ (resp. $\mathcal{H}^0_{\mathcal{A}}(d)$) is empty.

We assume that $\mathcal{A}$ is not $d$-trivial, otherwise, from (5.6), there is nothing to prove (the empty family of functors works for any $\mathcal{A}^{\mathbb{F}}$, and any finite field extension $\mathbb{F}$ of $\mathbb{L}$).

Consider the $\mathbb{L}$-constructible seminested ditalgebras $\mathcal{A}^{d_1}, \ldots, \mathcal{A}^{d_t}$ obtained from $\mathcal{A}$ by deletion of a finite number of idempotents of $R$. They are not almost sharply wild. Then, for any field extension $\mathbb{F}$ of $\mathbb{L}$, consider the $\mathbb{F}$-constructible seminested ditalgebras $\mathcal{A}^{\mathbb{F}d_1}, \dots, \mathcal{A}^{\mathbb{F}d_t}$ obtained from $\mathcal{A}^{\mathbb{F}}$ by deletion of the corresponding finite number of idempotents of $R^{\mathbb{F}}$. They are not almost sharply wild. Recall from [5, 20.4], that each $\mathcal{A}^{\mathbb{F}d_i}$ can be identified canonically with $\mathcal{A}^{d_i\mathbb{F}}$. We consider also $\mathcal{A}^{d_0} := \mathcal{A}$, $\mathcal{A}^{\mathbb{F}d_0} := \mathcal{A}^{\mathbb{F}}$, and the identity functor $F^{d_0} : \mathcal{A}^{\mathbb{F}d_0}\text{-Mod} \longrightarrow \mathcal{A}^{\mathbb{F}}\text{-Mod}$. Consider the subset $I$ of $[0, t]$ defined by $i \in I$ iff $\mathcal{A}^{d_i}$ is not sincerely $d$-trivial.

Then, apply (5.9)' to each $\mathcal{A}^{d_i}$ and $d$, for $i \in I$, to obtain finite field extensions $\{\mathbb{F}_i\}_{i\in I}$ of $\mathbb{L}$ satisfying the corresponding conditions. Then, if $\mathbb{F}_\omega$ denotes a finite field extension of $\mathbb{L}$ containing all $\mathbb{F}_i$, for $i \in I$, and $\mathbb{F}$ is any finite field extension of $\mathbb{F}_\omega$, there are functors $\{F_{i,j} : \Gamma_{i,j}\text{-Mod} \longrightarrow \mathcal{A}^{d_i\mathbb{F}}\text{-Mod}\}_{j=1}^{n_i}$ satisfying the corresponding statements *1–4* of the Corollary (5.9)' for each $\mathcal{A}^{d_i\mathbb{F}}$ and $d$. Then, we can consider the family of compositions

$$\mathcal{F}_{\mathbb{F}} := \left\{\Gamma_{i,j}\text{-Mod} \xrightarrow{F_{i,j}} \mathcal{A}^{d_i\mathbb{F}}\text{-Mod} = \mathcal{A}^{\mathbb{F}d_i}\text{-Mod} \xrightarrow{F^{d_i}} \mathcal{A}^{\mathbb{F}}\text{-Mod} \mid i \in I \text{ and } j \in [1, n_i]\right\}.$$

It is clear that the family $\mathcal{F}_{\mathbb{F}}$ satisfies item *1*, because the families $\{F_{i,j}\}_j$ do so and $F^{d_i}$ is sharp and endosharp. The family $\mathcal{F}_{\mathbb{F}}$ also satisfies *2* because given any $M \in \mathcal{M}_{\mathcal{A}^{\mathbb{F}}}(d)$, we have $M \cong F^{d_i}(N)$, for some $N \in \mathcal{M}^0_{\mathcal{A}^{\mathbb{F}d_i}}(d)$. For almost each one of these modules $N$, we have $F_{i,j}(H) \cong N$, for some $H \in \Gamma_{i,j}$-mod. Similarly, $\mathcal{F}_{\mathbb{F}}$ satisfies *3*. Item *4* follows from the application of [5, 22.7]. □

**Corollary 5.11.** *Let $\mathcal{A}$ be an $\mathbb{L}$-constructible seminested ditalgebra over a perfect field $k$. Then, $\mathcal{A}$ is centrally generically tame iff $\mathcal{A}$ is not almost sharply wild iff $\mathcal{A}$ is not almost endosharply wild.*

*Proof.* This follows from (4.9), (5.10), and (4.8). □

# 6 Tame and wild theorem

In the following paragraphs we establish the fundamental relation of the notion of centrally finite generic tameness with the notions of wildness introduced before: our Theorem (1.5). For technical reasons, we need to consider the following variation of the given notions of wildness.

**Definition 6.1.** A layered ditalgebra $\mathcal{A}$ is called *absolutely sharply wild* (resp. *absolutely endosharply wild*) iff there is some $A$-$k\langle x, y\rangle$-bimodule $Z$, free of finite rank by the right, such that the functor

$$\mathbb{F}\langle x, y\rangle\text{-Mod} \xrightarrow{Z^{\mathbb{F}} \otimes_{\mathbb{F}\langle x,y\rangle} -} \mathcal{A}^{\mathbb{F}}\text{-Mod} \xrightarrow{L_{\mathcal{A}^{\mathbb{F}}}} \mathcal{A}^{\mathbb{F}}\text{-Mod}$$

is sharp (resp. endosharp), for every finite field extension $\mathbb{F}$ of $k$. A layered ditalgebra $\mathcal{A}$ is called *almost absolutely sharply wild* (resp. *almost absolutely endosharply wild*) iff $\mathcal{A}^{\mathbb{F}}$ is absolutely sharply wild (resp. absolutely endosharply wild), for some finite field extension $\mathbb{F}$ of $k$.

**Lemma 6.2.** *For any non-zero polynomial $h \in k[x, y]$, the algebra $k[x, y]_h$ is almost absolutely sharply wild and almost absolutely endosharply wild.*

*Proof.* It is easy to show that there is a finite field extension $\mathbb{F}$ of $k$ and a polynomial $g \in \mathbb{F}[x,y]$ with $g(0,0) \neq 0$ such that for any finite field extension $\mathbb{E}$ of $\mathbb{F}$, we have $\mathbb{E}[x,y]_h \cong \mathbb{E}[x,y]_g$.

In order to see that $\mathbb{F}[x,y]_g$ is sharply and endosharply wild, we just have to notice that in the proof of [5, 22.16] the functor realizing the wildness of $\mathbb{F}[x,y]_g$ determines in fact a sharp and endosharp functor, see [5, 31.4].

We can show that $\mathbb{F}[x,y]_g$ is absolutely sharply wild (resp. absolutely endosharply wild) following the argument of the proof of [5, 30.7], where we have to notice that the bimodules which realize wildness for each scalar extension determine in fact sharp (resp. endosharp) functors. □

**Theorem 6.3.** *Every critical ditalgebra is absolutely sharply wild and absolutely endosharply wild.*

*Proof.* This is done as in the first part of the proof of [5](30.6), where the functors appearing there are sharp and endosharp. □

**Proposition 6.4.** *Given any almost sharply wild (or almost endosharply wild) $\mathbb{L}$-constructible seminested ditalgebra $\mathcal{A}$, over a perfect field, there is a finite field extension $\mathbb{F}$ of $\mathbb{L}$, a critical $\mathbb{F}$-ditalgebra $\mathcal{C}$, and functor $F : \mathcal{C}\text{-Mod} \longrightarrow \mathcal{A}^{\mathbb{F}}\text{-Mod}$, which is a composition of basic reduction functors of type $F^z$, with $z \in \{a,r,d,e,u\}$.*

*Proof.* This follows from the argument in the proof of (5.8), where applying basic reduction operations to any $\mathbb{L}$-constructible seminested ditalgebra $\mathcal{A}$, modulo finite field extensions, we either end up with minimal ditalgebras or at some stage we find a critical situation which is preserved under finite field extensions, see also (3.14) and its proof. If, for a given $d \in \mathbb{N}$, we end up with minimal ditalgebras over a finite field extension $\mathbb{F}$ of $\mathbb{L}$, this means that $\mathcal{A}^{\mathbb{F}}$ has finitely many non-isomorphic centrally finite endofinite indecomposables with central endolength bounded by $d$. From (4.8), the same holds for $\mathcal{A}$. Thus, $\mathcal{A}$ is centrally generically tame, and it is not almost sharply wild (and it is not almost endosharply wild). Thus, for an almost sharply wild (or an almost endosharply wild) ditalgebra $\mathcal{A}$ we must encounter the critical situation mentioned before. □

**Corollary 6.5.** *Any almost sharply wild (resp. almost endosharply wild) $\mathbb{L}$-constructible seminested ditalgebra $\mathcal{A}$ over a perfect field is almost absolutely sharply wild (resp. almost absolutely endosharply wild).*

*Proof.* From (6.4), there is a finite field extension $\mathbb{F}$ of $\mathbb{L}$, a critical $\mathbb{F}$-ditalgebra $\mathcal{C}$, and a functor $F : \mathcal{C}\text{-Mod} \longrightarrow \mathcal{A}^{\mathbb{F}}\text{-Mod}$, which is a composition of basic reduction functors of type $F^z$, with $z \in \{a,r,d,e,u\}$. From [5, §20], if $\mathcal{B}^z$ is the seminested ditalgebra obtained from a seminested $\mathbb{F}$-ditalgebra $\mathcal{B}$ by a basic operation of type $z \in \{a,r,d,e,u\}$, and $\mathbb{E}$ is any finite field extension of $\mathbb{F}$, we can identify $\mathcal{B}^{z\mathbb{E}}$ with $\mathcal{B}^{\mathbb{E}z}$ and the following diagram commutes up to isomorphism

$$\begin{array}{ccc} \mathcal{B}^{\mathbb{E}z}\text{-Mod} & \xrightarrow{F_z^{\mathbb{E}}} & \mathcal{B}^{\mathbb{E}}\text{-Mod} \\ {\scriptstyle (-)^{\mathbb{E}}}\uparrow & & \uparrow{\scriptstyle (-)^{\mathbb{E}}} \\ \mathcal{B}^{z}\text{-Mod} & \xrightarrow{F_z} & \mathcal{B}\text{-Mod}, \end{array}$$

where $F_z^{\mathbb{E}}$ is the functor associated to the basic operation $\mathcal{B}^{\mathbb{E}} \mapsto \mathcal{B}^{\mathbb{E}z}$. So we denote by $F^{\mathbb{E}}$ the composition of the functors $F_z^{\mathbb{E}}$ corresponding the functors $F_z$ appearing in the composition $F$. Then, we also get a functor $F^{\mathbb{E}}$ : $\mathcal{C}^{\mathbb{E}}\text{-Mod}\longrightarrow\mathcal{A}^{\mathbb{E}}\text{-Mod}$, where $\mathcal{C}^{\mathbb{E}}$ is a critical $\mathbb{E}$-ditalgebra. Thus, the functor $F^{\mathbb{E}}$ is full and faithful and preserves endofinite modules, and hence it is a sharp and endosharp functor.

By (6.3), applied to the critical ditalgebra $\mathcal{C}$, there is a $C^{\mathbb{F}}$-$\mathbb{F}\langle x, y\rangle$-bimodule $Z$, free of finite rank by the right, such that for each finite field extension $\mathbb{E}$ of $\mathbb{F}$, the following functor is sharp (resp. endosharp):

$$\mathbb{E}\langle x,y\rangle\text{-Mod}\xrightarrow{Z^{\mathbb{E}}\otimes_{\mathbb{E}\langle x,y\rangle}-}C^{\mathbb{E}}\text{-Mod}\xrightarrow{L_{\mathcal{C}^{\mathbb{E}}}}\mathcal{C}^{\mathbb{E}}\text{-Mod}.$$

From [5, 22.7], the following functor is also sharp (resp. endosharp)

$$\mathbb{E}\langle x,y\rangle\text{-Mod}\xrightarrow{F^{\mathbb{E}}(Z^{\mathbb{E}})\otimes_{\mathbb{E}\langle x,y\rangle}-}A^{\mathbb{E}}\text{-Mod}\xrightarrow{L_{\mathcal{A}^{\mathbb{E}}}}\mathcal{A}^{\mathbb{E}}\text{-Mod},$$

and the bimodule $F^{\mathbb{E}}(Z^{\mathbb{E}})$ is finitely generated projective by the right (hence free of finite rank). The commutativities of the above diagrams give $F^{\mathbb{E}}(Z^{\mathbb{E}}) \cong F(Z)^{\mathbb{E}}$, so we get that the $A^{\mathbb{F}}$-$\mathbb{F}\langle x, y\rangle$-bimodule $F(Z)$ realizes the absolute sharp wildness (resp. absolute endosharp wildness) of $\mathcal{A}^{\mathbb{F}}$. □

**Lemma 6.6.** *Consider a finite-dimensional algebra $\Lambda$, over a perfect field $k$, and its Drozd's ditalgebra $\mathcal{D} = \mathcal{D}^{\Lambda}$. If $\mathcal{D}$ is sharply (resp. endosharply) wild, then the following composition functor is sharp (resp. endosharp)*

$$k\langle x,y\rangle\text{-Mod}\xrightarrow{Z\otimes_{k\langle x,y\rangle}-}D\text{-Mod}\xrightarrow{L_{\mathcal{D}}}\mathcal{D}\text{-Mod}\xrightarrow{\Xi_{\Lambda}}\mathcal{P}^1(\Lambda)\xrightarrow{\text{Cok}}\Lambda\text{-Mod}.$$

*Here, the bimodule $Z$ realizes the sharp (resp. endosharp) wildness of $\mathcal{D}$ and is given by our assumption, $\Xi_{\Lambda}$ denotes the usual equivalence, and* Cok *is the cokernel functor.*

*Proof.* Since $k\langle x, y\rangle$ admits an infinite number of non-isomorphic finite-dimensional (hence endofinite) indecomposables, then we can use the same argument of the proof of [5, 22.20(1)] and [2, 4.4], to guarantee that the composition preserves indecomposables (resp. endofinite indecomposables). Using [5, 18.10] and the fact that the functor $L_{\mathcal{D}}(Z \otimes_{k\langle x,y\rangle} -)$ is sharp (resp. endosharp), we obtain that the above composition functor is sharp (resp. endosharp). □

**Lemma 6.7.** *Consider a finite-dimensional algebra $\Lambda$ over a perfect field $k$, and suppose that its Drozd's ditalgebra $\mathcal{D}^{\Lambda}$ is semielementary. Then,*

1. *$\mathcal{D}^{\Lambda b}$ is centrally generically tame iff $\mathcal{D}^{\Lambda}$ is so;*

2. $\mathcal{D}^{\Lambda b}$ *is almost sharply wild (resp. almost endosharply wild) iff* $\mathcal{D}^{\Lambda}$ *is so;*

3. $\mathcal{D}^{\Lambda b}$ *is almost absolutely sharply wild (resp. almost absolutely endosharply wild) iff* $\mathcal{D}^{\Lambda}$ *is so.*

*Proof.* Consider the basification equivalence functor $F^b : \mathcal{D}^{\Lambda b}\text{-Mod}\longrightarrow\mathcal{D}^{\Lambda}\text{-Mod}$, as in [2, 3.3]. Then $F^b$ is a sharp endosharp functor and (1) follows from [2, 3.3]. We denote by $\hat{F}^b : \mathcal{D}^{\Lambda\mathbb{F}b}\text{-Mod}\longrightarrow\mathcal{D}^{\Lambda\mathbb{F}}\text{-Mod}$ the basification equivalence functor, for any finite field extension $\mathbb{F}$ of $k$. Recall that we can identify $(\mathcal{D}^{\Lambda})^{\mathbb{F}}$ with $\mathcal{D}^{\Lambda\mathbb{F}}$, and $\mathcal{D}^{\Lambda\mathbb{F}b}$ with $\mathcal{D}^{\Lambda b\mathbb{F}}$.

(2) If $\mathcal{D}^{\Lambda b\mathbb{F}} = \mathcal{D}^{\Lambda\mathbb{F}b}$ is sharply wild (resp. endosharply wild), for some finite field extension $\mathbb{F}$ of $k$, from [5, 22.7], we know that if $\mathcal{D}^{\Lambda\mathbb{F}}$ is sharply wild (resp. endosharply wild).

Now, assume that $\mathcal{D}^{\Lambda\mathbb{F}}$ is sharply wild (resp. endosharply wild), for some finite field extension $\mathbb{F}$ of $k$. Then, by (3.9), $\mathcal{D}^{\Lambda\mathbb{F}}$ is not centrally generically tame. By assumption, $\mathcal{D}^{\Lambda}$ is semielementary. Hence, $\mathcal{D}^{\Lambda\mathbb{F}}$ is semielementary. From item *1*, we obtain that $\mathcal{D}^{\Lambda\mathbb{F}b}$ is not centrally generically tame. By (5.11), the $\mathbb{F}$-constructible seminested ditalgebra $\mathcal{D}^{\Lambda\mathbb{F}b} = \mathcal{D}^{\Lambda b\mathbb{F}}$ is almost sharply wild (resp. almost endosharply wild). Then, $\mathcal{D}^{\Lambda b}$ is almost sharply wild (resp. almost endosharply wild).

(3) Assume that $\mathcal{D}^{\Lambda b}$ is almost absolutely sharply wild (resp. almost absolutely endosharply wild). Then, there is a finite field extension $\mathbb{F}$ of $k$, and some $D^{\Lambda b\mathbb{F}}$-$\mathbb{F}\langle x,y\rangle$-bimodule $B$, free of finite rank by the right, such that the functor

$$\mathbb{E}\langle x,y\rangle\text{-Mod}\xrightarrow{B^{\mathbb{E}}\otimes_{\mathbb{E}\langle x,y\rangle}-}D^{\Lambda b\mathbb{E}}\text{-Mod}\xrightarrow{L_{\mathcal{D}^{\Lambda b\mathbb{E}}}}\mathcal{D}^{\Lambda b\mathbb{E}}\text{-Mod}$$

is sharp (resp. endosharp), for every finite field extension $\mathbb{E}$ of $\mathbb{F}$. Then, by [5, 22.7], the composition functor

$$\mathbb{F}\langle x,y\rangle\text{-Mod}\xrightarrow{B\otimes_{\mathbb{F}\langle x,y\rangle}-}D^{\Lambda b\mathbb{F}}\text{-Mod}\xrightarrow{L_{\mathcal{D}^{\Lambda b\mathbb{F}}}}\mathcal{D}^{\Lambda b\mathbb{F}}\text{-Mod}\xrightarrow{F^b}\mathcal{D}^{\Lambda\mathbb{F}}\text{-Mod}$$

is a sharp (resp. endosharp) and naturally isomorphic to $L_{\mathcal{D}^{\Lambda\mathbb{F}}}(F^b(B)\otimes_{\mathbb{F}\langle x,y\rangle}-)$, where the $D^{\Lambda\mathbb{F}}$-$\mathbb{F}\langle x,y\rangle$-bimodule $F^b(B)$ is free of finite rank by the right. We claim that $L_{\mathcal{D}^{\Lambda\mathbb{E}}}(F^b(B)^{\mathbb{E}}\otimes_{\mathbb{E}\langle x,y\rangle}-)$ is sharp (resp. endosharp), for any finite field extension $\mathbb{E}$ of $\mathbb{F}$. Indeed, the following composition is also sharp (resp. endosharp)

$$\mathbb{E}\langle x,y\rangle\text{-Mod}\xrightarrow{B^{\mathbb{E}}\otimes_{\mathbb{E}\langle x,y\rangle}-}D^{\Lambda b\mathbb{E}}\text{-Mod}\xrightarrow{L_{\mathcal{D}^{\Lambda b\mathbb{E}}}}\mathcal{D}^{\Lambda b\mathbb{E}}\text{-Mod}\xrightarrow{\hat{F}^b}\mathcal{D}^{\Lambda\mathbb{E}}\text{-Mod}$$

and is naturally isomorphic to $L_{\mathcal{D}^{\Lambda b\mathbb{E}}}(\hat{F}^b(B^{\mathbb{E}})\otimes_{\mathbb{E}\langle x,y\rangle}-)$, where the $\Lambda^{\mathbb{E}}$-$\mathbb{E}\langle x,y\rangle$-bimodule $\hat{F}^b(B^{\mathbb{E}})\cong F^b(B)^{\mathbb{E}}$ is free of finite rank by the right. Thus, $\mathcal{D}^{\Lambda}$ is almost absolutely sharply wild (resp. almost absolutely endosharply wild).

If $\mathcal{D}^{\Lambda}$ is almost absolutely sharply wild, it is almost sharply wild. Thus, item *2* implies that $\mathcal{D}^{\Lambda b}$ is almost sharply wild. Therefore, by (6.5), $\mathcal{D}^{\Lambda b}$ is almost absolutely sharply wild. The argument for almost absolute endosharp wildness is the same. □

**Proposition 6.8.** *Let $\Lambda$ be a finite-dimensional algebra over a perfect field $k$. Then, if $\mathcal{D}^\Lambda$ is almost absolutely sharply wild (resp. almost absolutely endosharply wild) then $\Lambda$ is almost absolutely sharply wild (resp. almost absolutely endosharply wild).*

*Proof.* Assume that $\mathcal{D}^\Lambda$ is almost absolutely sharply wild (resp. almost absolutely endosharply wild). Then, there is a finite field extension $\mathbb{F}$ of $k$, and some $D^{\Lambda^\mathbb{F}}$-$\mathbb{F}\langle x,y\rangle$-bimodule $B$, free of finite rank by the right, such that the functor

$$\mathbb{E}\langle x,y\rangle\text{-Mod}\xrightarrow{B^\mathbb{E}\otimes_{\mathbb{E}\langle x,y\rangle}-}D^{\Lambda^\mathbb{E}}\text{-Mod}\xrightarrow{L_{\mathcal{D}^{\Lambda^\mathbb{E}}}}\mathcal{D}^{\Lambda^\mathbb{E}}\text{-Mod}$$

is sharp (resp. endosharp), for every finite field extension $\mathbb{E}$ of $\mathbb{F}$. By (6.6) and [5, 22.18], the composition

$$\mathbb{F}\langle x,y\rangle\text{-Mod}\xrightarrow{B\otimes_{\mathbb{F}\langle x,y\rangle}-}D^{\Lambda^\mathbb{F}}\text{-Mod}\xrightarrow{L_{\mathcal{D}^{\Lambda^\mathbb{F}}}}\mathcal{D}^{\Lambda^\mathbb{F}}\text{-Mod}\xrightarrow{\Xi}\mathcal{P}^1(\Lambda^\mathbb{F})\xrightarrow{\text{Cok}}\Lambda^\mathbb{F}\text{-Mod}$$

is a sharp (resp. endosharp) functor naturally isomorphic to $Z\otimes_{\mathbb{F}\langle x,y\rangle}-$, where the $\Lambda^\mathbb{F}$-$\mathbb{F}\langle x,y\rangle$-bimodule $Z=\text{Cok}\Xi(B)$ is finitely generated (may be not free) by the right. We claim that $Z^\mathbb{E}\otimes_{\mathbb{E}\langle x,y\rangle}-$ is sharp (resp. endosharp), for any finite field extension $\mathbb{E}$ of $\mathbb{F}$. Indeed, by (6.6) and [5, 22.18], the following composition is also sharp (resp. endosharp)

$$\mathbb{E}\langle x,y\rangle\text{-Mod}\xrightarrow{B^\mathbb{E}\otimes_{\mathbb{E}\langle x,y\rangle}-}D^{\Lambda^\mathbb{E}}\text{-Mod}\xrightarrow{L_{\mathcal{D}^{\Lambda^\mathbb{E}}}}\mathcal{D}^{\Lambda^\mathbb{E}}\text{-Mod}\xrightarrow{\Xi}\mathcal{P}^1(\Lambda^\mathbb{E})\xrightarrow{\text{Cok}}\Lambda^\mathbb{E}\text{-Mod}$$

and is naturally isomorphic to $Z^\mathbb{E}\otimes_{\mathbb{E}\langle x,y\rangle}-$, where the $\Lambda^\mathbb{E}$-$\mathbb{E}\langle x,y\rangle$-bimodule $Z^\mathbb{E}\cong\text{Cok}\Xi(B^\mathbb{E})$ is finitely generated (may be not free) by the right (indeed, the functors Cok and $\Xi$ commute with scalar extensions).

Now, consider the functor $\mathbb{F}[x,y]\otimes_{\mathbb{F}[x,y]}-:\mathbb{F}[x,y]\text{-Mod}\longrightarrow\mathbb{F}\langle x,y\rangle\text{-Mod}$ and the $\Lambda^\mathbb{F}$-$\mathbb{F}[x,y]$-bimodule $Z_1:=Z\otimes_{\mathbb{F}\langle x,y\rangle}\mathbb{F}[x,y]$. Hence, for each finite field extension $\mathbb{E}$ of $\mathbb{F}$, we have a sharp (resp. endosharp) functor $Z_1^\mathbb{E}\otimes_{\mathbb{E}[x,y]}-:\mathbb{E}[x,y]\text{-Mod}\longrightarrow\Lambda^\mathbb{E}\text{-Mod}$.

Now, consider a presentation of the $\mathbb{F}[x,y]$-module $Z_1$

$$\mathbb{F}[x,y]^s\xrightarrow{H}\mathbb{F}[x,y]^r\longrightarrow Z_1\longrightarrow 0.$$

Then, there are invertible matrices $P$ and $Q$ with entries in $\mathbb{F}[x,y]_h$, for some non-zero polynomial $h\in\mathbb{F}[x,y]$, such that $PHQ=\begin{pmatrix}I&0\\0&0\end{pmatrix}$. This implies that the $\Lambda^\mathbb{F}$-$\mathbb{F}[x,y]_h$-bimodule $Z_2:=Z_1\otimes_{\mathbb{F}[x,y]}\mathbb{F}[x,y]_h$ is free of finite rank by the right, see the proof of [5, 22.17].

The same matrices $P,H,Q$ have entries in $\mathbb{E}[x,y]_h$, for any finite field extension $\mathbb{E}$ of $\mathbb{F}$, and we still have a presentation of the $\mathbb{E}[x,y]$-module $Z_1^\mathbb{E}$

$$\mathbb{E}[x,y]^s\xrightarrow{H}\mathbb{E}[x,y]^r\longrightarrow Z_1^\mathbb{E}\longrightarrow 0.$$

Again, this implies that the $\Lambda^\mathbb{E}$-$\mathbb{E}[x,y]_h$-bimodule $Z_2^\mathbb{E}\cong Z_1^\mathbb{E}\otimes_{\mathbb{E}[x,y]}\mathbb{E}[x,y]_h$ is free of finite rank by the right.

Moreover, the functor $Z_2^{\mathbb{E}} \otimes_{\mathbb{E}[x,y]_h} - : \mathbb{E}[x,y]_h\text{-Mod} \longrightarrow \Lambda^{\mathbb{E}}\text{-Mod}$ is sharp (resp. endosharp).

Now, we know from (6.2) that $\mathbb{F}[x,y]_h$ is almost absolutely sharply wild (resp. absolutely endosharply wild). Then, there is a finite field extension $\mathbb{F}_w$ of $\mathbb{F}$ such that $\mathbb{F}_w[x,y]_h$ is sharply wild (resp. endosharply wild), say with the bimodule $X$, then we have that $Z_3 := Z_2^{\mathbb{F}_w} \otimes_{\mathbb{F}_w[x,y]_h} X$ is a $\Lambda^{\mathbb{F}_w}$-$\mathbb{F}_w\langle x,y\rangle$-bimodule, free of finite rank by the right, such that, for any finite field extension $\mathbb{E}$ of $\mathbb{F}_w$, the functor $Z_3^{\mathbb{E}} \otimes_{\mathbb{E}\langle x,y\rangle} - : \mathbb{E}\langle x,y\rangle\text{-Mod} \longrightarrow \Lambda^{\mathbb{E}}\text{-Mod}$ is sharp (resp. endosharp). □

**Lemma 6.9.** *Let $\mathbb{L}$ be a finite field extension of the perfect field $k$ and $\Lambda$ a finite-dimensional $k$-algebra. Then, for any $d \in \mathbb{N}$, there is only a finite number of isoclasses of centrally finite generic $\Lambda^{\mathbb{L}}$-modules $H$ with $c\text{-endol}\,(H) \leq d$ iff there is only a finite number of isoclasses of centrally finite generic $\Lambda$-modules $G$ with $c\text{-endol}\,(G) \leq d$. Thus, $\Lambda^{\mathbb{L}}$ is centrally generically tame iff $\Lambda$ is centrally generically tame.*

*Proof.* The same proof given in [15, 2.19] works here. □

**Theorem 6.10.** *Let $\Lambda$ be a finite-dimensional algebra over a perfect field. Then the following statements are equivalent:*

1. *$\Lambda$ is not centrally generically tame;*
2. *$\Lambda$ is almost sharply wild;*
3. *$\Lambda$ is almost absolutely sharply wild;*
4. *$\Lambda$ is almost endosharply wild;*
5. *$\Lambda$ is almost absolutely endosharply wild.*

*Proof.* Consider a finite field extension $\mathbb{L}$ of $k$ such that $\Lambda^{\mathbb{L}}$ admits a splitting $\Lambda^{\mathbb{L}} = S \oplus \mathrm{rad}\Lambda^{\mathbb{L}}$, where the semisimple $S$ is a finite product of matrix algebras with coefficients in $\mathbb{L}$, or, equivalently, such that $\mathcal{D}^{\Lambda\mathbb{L}}$ is semielementary (and $\mathcal{D}^{\Lambda\mathbb{L}b}$ is constructible).

Obviously, *3* implies *2* and *5* implies *4*. By (3.9) and (6.9), we know that *2* implies *1*, and *4* implies *1*.

In order to show that *1* implies *3* and *5*, assume that $\Lambda$ is not centrally generically tame. From (6.9), $\Lambda^{\mathbb{L}}$ is not centrally generically tame. By (2.13), $\mathcal{D}^{\Lambda\mathbb{L}}$ is not centrally generically tame. Then, (6.7) implies that $\mathcal{D}^{\Lambda\mathbb{L}b}$ is not centrally generically tame, and (5.11) implies that $\mathcal{D}^{\Lambda\mathbb{L}b}$ is almost sharply wild and almost endosharply wild. By (6.5), $\mathcal{D}^{\Lambda\mathbb{L}b}$ is almost absolutely sharply wild and almost absolutely endosharply wild. Hence, (6.7) implies that this holds for $\mathcal{D}^{\Lambda\mathbb{L}}$. By (6.8), $\Lambda^{\mathbb{L}}$ is almost absolutely sharply wild and almost absolutely endosharply wild. Hence, *3* and *5* hold for $\Lambda$. □

# 7 Parametrization theorem

In this last section we transfer the parametrization theorem (5.10), for modules over constructible seminested ditalgebras, to modules over finite-dimensional algebras over perfect fields. This will be used to derive our theorem (1.6).

**Theorem 7.1.** *Let $\Lambda$ be a centrally generically tame finite-dimensional algebra over a (possibly finite) perfect field $k$, and let $d$ be a non-negative integer. Then, there is a finite field extension $\mathbb{F}_\omega$ of $k$ such that: for any finite field extension $\mathbb{F}$ of $\mathbb{F}_\omega$, there are rational $\mathbb{F}$-algebras $\Gamma_1, \ldots, \Gamma_m$, and $\Lambda^{\mathbb{F}}$-$\Gamma_i$-bimodules $Z_1, \ldots, Z_m$, which are finitely generated as right $\Gamma_i$-modules, satisfying the following:*

1. *The functor $Z_i \otimes_{\Gamma_i} - : \Gamma_i\text{-Mod} \longrightarrow \Lambda^{\mathbb{F}}\text{-Mod}$ is sharp and $Z_i \otimes_{\Gamma_i} \mathbb{F}(x)$ is an algebraically rigid centrally finite generic $\Lambda^{\mathbb{F}}$-module, for $i \in [1, m]$;*
2. *For almost every finite-dimensional indecomposable $M \in \Lambda^{\mathbb{F}}\text{-Mod}$ with $c\text{-endol}\,(M) \leq d$, there are $i \in [1, m]$ and $N \in \Gamma_i\text{-Mod}$ with $Z_i \otimes_{\Gamma_i} N \cong M$ in $\Lambda^{\mathbb{F}}\text{-Mod}$;*
3. *For every centrally finite generic $G \in \Lambda^{\mathbb{F}}\text{-Mod}$ with $c\text{-endol}\,(G) \leq d$, there is a unique $i \in [1, m]$ with $Z_i \otimes_{\Gamma_i} \mathbb{F}(x) \cong G$ in $\Lambda^{\mathbb{F}}\text{-Mod}$.*

*Proof.* Let $\Lambda$ be a centrally generically tame finite-dimensional algebra over a perfect field $k$ and let $d \in \mathbb{N}$. Since $k$ is a perfect field, there is a finite field extension $\mathbb{L}$ of $k$ such that the Drozd's ditalgebra of $\Lambda^{\mathbb{L}}$ is semielementary. By (6.9), $\Lambda^{\mathbb{L}}$ is centrally generically tame. Then, $\mathcal{D}^{\Lambda\mathbb{L}b}$ is an $\mathbb{L}$-constructible seminested ditalgebra not almost sharply wild, by (2.13) and (6.7). Moreover, for each finite field extension $\mathbb{F}$ of $\mathbb{L}$, we can identify $\mathcal{D}^{\Lambda^{\mathbb{F}}}$ with $\mathcal{D}^{\Lambda\mathbb{F}}$, see [5, 20.13], and consider the basification equivalence functor $F^b : \mathcal{D}^{\Lambda\mathbb{F}b}\text{-Mod} \longrightarrow \mathcal{D}^{\Lambda\mathbb{F}}\text{-Mod}$. From [4, 2.11 and 2.12], there is a positive integer $\mu_b$, independent of the field $\mathbb{F}$, such that $c\text{-endol}\,(N) \leq \mu_b \times c\text{-endol}\,(F_b(N))$, for any centrally finite $N \in \mathcal{D}^{\Lambda\mathbb{F}b}\text{-Mod}$.

Make $d' := (1 + \dim_k \Lambda) \times d$ and $d'' := \mu_b \times d'$.

Apply (5.10) to the ditalgebra $\mathcal{D}^{\Lambda\mathbb{L}b}$ and the integer $d''$ to obtain a finite field extension $\mathbb{F}_\omega$ of $\mathbb{L}$ such that, for any finite field extension $\mathbb{F}$ of $\mathbb{F}_\omega$, there is a family of functors $F_i : \Gamma_i\text{-Mod} \longrightarrow \mathcal{D}^{\Lambda\mathbb{F}b}\text{-Mod}$, $i \in [1, m]$, such that *1–4* of (5.10) hold for $\mathcal{D}^{\Lambda\mathbb{F}b}$ and $d''$ (recall that we can identify $\mathcal{D}^{\Lambda\mathbb{L}b\mathbb{F}}$ with $\mathcal{D}^{\Lambda\mathbb{F}b}$, by [5, 20.11]).

For any $i \in [1, m]$, we have $F_i \cong L_{\mathcal{D}^{\Lambda\mathbb{F}b}}(Y_i \otimes_{\Gamma_i} -)$, where $Y_i$ is a $D^{\Lambda\mathbb{F}b}$-$\Gamma_i$-bimodule, free of finite rank as a right $\Gamma_i$-module and $F_i$ is sharp, and preserves pregeneric modules. From [5, 22.7], we get that $F^b F_i \cong L_{\mathcal{D}^{\Lambda\mathbb{F}}}(F^b(Y_i) \otimes_{\Gamma_i} -)$, where $F^b(Y_i)$ is a $D^{\Lambda\mathbb{F}}$-$\Gamma_i$-bimodule, free of finite rank as a right $\Gamma_i$-module and $F^b F_i$ is sharp, and preserves pregeneric modules.

Consider the usual equivalence functor $\Xi_{\Lambda^{\mathbb{F}}} : \mathcal{D}^{\Lambda\mathbb{F}}\text{-Mod} \longrightarrow \mathcal{P}^1(\Lambda^{\mathbb{F}})$ and, for $i \in [1, m]$, set $Z_i := Z \otimes_{D^{\Lambda\mathbb{F}}} F^b(Y_i)$, where $Z$ is the transition bimodule associated to $\Lambda^{\mathbb{F}}$, as in [5, 22.18]. Each bimodule $Z_i$ is finitely generated over $\Gamma_i$ by construction. For each $i$, denote by $U_i$ the composition

$$\Gamma_i\text{-Mod} \xrightarrow{F^bF_i} \mathcal{D}^{\Lambda^{\mathbb{F}}}\text{-Mod} \xrightarrow{\Xi_{\Lambda^{\mathbb{F}}}} \mathcal{P}^1(\Lambda^{\mathbb{F}}) \xrightarrow{\mathrm{Cok}} \Lambda^{\mathbb{F}}\text{-Mod}.$$

Then, $U_i \cong \mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}} L_{\mathcal{D}^{\Lambda^{\mathbb{F}}}}(F^b(Y_i) \otimes_{\Gamma_i} -) \cong Z \otimes_{D^{\Lambda^{\mathbb{F}}}} F^b(Y_i) \otimes_{\Gamma_i} - = Z_i \otimes_{\Gamma_i} -$.

The functor $L_{\mathcal{D}^{\Lambda^{\mathbb{F}}}}(F^b(Y_i) \otimes_{\Gamma_i} -)$ preserves isomorphism classes of indecomposables. From [5, 22.20], we know that $\Xi_{\Lambda^{\mathbb{F}}} F^b F_i \cong \Xi_{\Lambda^{\mathbb{F}}} L_{\mathcal{D}^{\Lambda^{\mathbb{F}}}}(F^b(Y_i) \otimes_{\Gamma_i} -)$ is a sharp functor which maps indecomposable $\Gamma_i$-modules into $\mathcal{P}^2(\Lambda^{\mathbb{F}})$. By [5, 18.10], the functor $\mathrm{Cok} : \mathcal{P}^2(\Lambda^{\mathbb{F}}) \longrightarrow \Lambda^{\mathbb{F}}$-Mod is also sharp, and then the same holds for $U_i$. The last statement of *1* follows from property (5.10)(1) for $F_i$, [4, 2.11], and (2.12).

(2): Take a finite-dimensional indecomposable $\Lambda^{\mathbb{F}}$-module $M$ satisfying that $c$-endol$(M) \leq d$ and a module $L \in \mathcal{D}^{\Lambda^{\mathbb{F}}}$-Mod with $\Xi_{\Lambda^{\mathbb{F}}}(L) \in \mathcal{P}^2(\Lambda^{\mathbb{F}})$ and $\mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}}(L) \cong M$. Then, take $L' \in \mathcal{D}^{\Lambda^{\mathbb{F}}b}$-Mod with $F^b(L') \cong L$. From (2.12), the finite-dimensional indecomposable $\mathcal{D}^{\Lambda^{\mathbb{F}}}$-module $L$ satisfies that $c$-endol$(L) \leq d'$, and $L' \in \mathcal{M}_{\mathcal{D}^{\Lambda^{\mathbb{F}}b}}(d'')$. From (5.10)(2), we know that for almost every such modules $L'$, we have that $L' \cong F_i(N)$, for some $i \in [1, m]$ and $N \in \Gamma_i$-mod. Hence, $M \cong \mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}}(L) \cong \mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}} F^b F_i(N) \cong Z_i \otimes_{\Gamma_i} N$.

(3): Take a centrally finite generic $\Lambda^{\mathbb{F}}$-module $G$ with $c$-endol$(G) \leq d$ and a module $H \in \mathcal{D}^{\Lambda^{\mathbb{F}}}$-Mod with $\Xi_{\Lambda^{\mathbb{F}}}(H) \in \mathcal{P}^2(\Lambda^{\mathbb{F}})$ and $\mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}}(H) \cong G$. Then, take $H' \in \mathcal{D}^{\Lambda^{\mathbb{F}}b}$-Mod with $F^b(H') \cong H$. From (2.12), $H$ is a centrally finite pregeneric $\mathcal{D}^{\Lambda^{\mathbb{F}}}$-module with $c$-endol$(H) \leq d'$, and $H' \in \mathcal{H}_{\mathcal{D}^{\Lambda^{\mathbb{F}}b}}(d'')$. From (5.10)(3), we have $H' \cong F_i(\mathbb{F}(x))$ in $\mathcal{D}^{\Lambda^{\mathbb{F}}}$-Mod, for $i \in [1, m]$. Thus, $G \cong \mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}}(H) \cong \mathrm{Cok}\Xi_{\Lambda^{\mathbb{F}}} F^b F_i(\mathbb{F}(x)) \cong Z_i \otimes_{\Gamma_i} \mathbb{F}(x)$. □

**Lemma 7.2.** *Let $\Lambda$ be a finite-dimensional algebra over a perfect field $k$, $d \in \mathbb{N}$, and $\mathbb{L}$ a finite field extension of $k$. If every centrally finite generic $\Lambda^{\mathbb{L}}$-module $H$ with $c$-*endol$(H) \leq d$ *is algebraically bounded, then every centrally finite generic $\Lambda$-module $G$ with $c$-*endol$(G) \leq d$ *is algebraically bounded.*

*Proof.* Let $G$ be a centrally finite generic $\Lambda$-module with $c$-endol$(G) \leq d$. From [15, 2.14], we know that $G^{\mathbb{L}} \cong m_1 G_1 \bigoplus \cdots \bigoplus m_t G_t$, where $G_1, \ldots, G_t$ are generic $\Lambda^{\mathbb{L}}$-modules, and $G$ is algebraically bounded iff $G_j$ is algebraically bounded for some $j \in [1, t]$. But, from [15, 2.18], $c$-endol$(G) = c$-endol$(G_i)$, for all $i$, and there is $j \in [1, t]$ such that $G_j$ is centrally finite. By assumption, $G_j$ is algebraically bounded and we are done. □

The content and proof of our theorem (1.6) is divided in the following two theorems.

**Theorem 7.3.** *Assume that $\Lambda$ is a finite-dimensional algebra over a perfect field. If $\Lambda$ is centrally generically tame and $G$ is a generic $\Lambda$-module, then $G$ is centrally finite iff $G$ is algebraically bounded.*

*Proof.* Assume that $\Lambda$ is centrally generically tame. Therefore, given $d \in \mathbb{N}$, there is a finite field extension $\mathbb{L}$ of $k$ such that the centrally finite generic $\Lambda^{\mathbb{L}}$-modules $G$ with $c$-endol$(G) \leq d$ can be parametrized as described in (7.1). Then, any such centrally finite generic $\Lambda^{\mathbb{L}}$-module $G$ is algebraically bounded, by item *1* of (7.1). By (7.2), any centrally finite generic $\Lambda$-module $H$ with $c$-endol $(H) \leq d$ is algebraically bounded. Since this holds for each $d \in \mathbb{N}$, any centrally finite generic $\Lambda$-module is algebraically bounded. Finally, it is clear that $\Lambda$ is semigenerically tame and, from [15, 1.8], we know that algebraically bounded generic $\Lambda$-modules are centrally finite. □

In the proof of our next theorem, given a field $k$, we denote by $U(k\langle x,y\rangle)$ the universal field of fractions of the free ideal $k$-algebra $k\langle x,y\rangle$, see [8]. Then, for instance, by Bergman's centralizer theorem (see [6]), the center of $U(k\langle x,y\rangle)$ is $k$, and the next observation follows from [9, Corollary 1.1].

**Lemma 7.4.** *For any field extension $\mathbb{E}$ of $k$ there is an isomorphism of $\mathbb{E}$-algebras*

$$(U(k\langle x,y\rangle))^{\mathbb{E}} \cong U(\mathbb{E}\langle x,y\rangle).$$

**Theorem 7.5.** *Let $\Lambda$ be a finite-dimensional algebra over a perfect field $k$. Then, $\Lambda$ is centrally generically tame iff $\Lambda$ is semigenerically tame.*

*Proof.* Assume that $\Lambda$ is semigenerically tame but not centrally generically tame. By theorem (6.10), $\Lambda$ is almost absolutely sharply wild. Then, we have a finite field extension $\mathbb{F}$ of $k$, a $\Lambda^{\mathbb{F}}$-$\mathbb{F}\langle x,y\rangle$-bimodule $Z$, which is free of finite rank by the right, such that for each finite field extension $\mathbb{E}$ of $\mathbb{F}$ we get a sharp functor

$$Z^{\mathbb{E}} \otimes_{\mathbb{E}\langle x,y\rangle} - : \mathbb{E}\langle x,y\rangle\text{-Mod} \longrightarrow \Lambda^{\mathbb{E}}\text{-Mod}.$$

Consider the $\mathbb{F}\langle x,y\rangle$-module $H := U(\mathbb{F}\langle x,y\rangle)$. It is clear that $H$ is an endofinite indecomposable, because the canonical embedding $\sigma : \mathbb{F}\langle x,y\rangle \longrightarrow U(\mathbb{F}\langle x,y\rangle)$ is an epimorphism and so it induces by restriction a full and faithful functor $F_\sigma : U(\mathbb{F}\langle x,y\rangle)\text{-Mod} \longrightarrow \mathbb{F}\langle x,y\rangle\text{-Mod}$. Thus, the $\mathbb{F}\langle x,y\rangle$-module $H$ is pregeneric. Moreover, by Bergman's centralizer theorem, the center of the skew field $U(\mathbb{F}\langle x,y\rangle) \cong \mathrm{End}_{U(\mathbb{F}\langle x,y\rangle)}(U(\mathbb{F}\langle x,y\rangle)) \cong \mathrm{End}_{\mathbb{F}\langle x,y\rangle}(H)$ is $\mathbb{F}$, so $D_H \cong U(\mathbb{F}\langle x,y\rangle)$ has infinite dimension over its center $\mathbb{F}$, and $H$ is not centrally finite.

Then, $G := Z \otimes_{\mathbb{F}\langle x,y\rangle} H$ is a generic $\Lambda^{\mathbb{F}}$-module as observed by Crawley-Boevey in [12, 7.4, proof of Corollary], see also [5, 31.4]. But, since $Z \otimes_{\mathbb{F}(\langle x,y\rangle)} -$ is a sharp functor and $H$ is not centrally finite, the generic $\Lambda^{\mathbb{F}}$-module $G$ is not centrally finite.

It will be enough to show that $G$ is an algebraically bounded generic $\Lambda^{\mathbb{F}}$-module. This will contradict the fact that every algebraically bounded generic module over the semigenerically tame finite-dimensional algebra $\Lambda^{\mathbb{F}}$ is centrally finite, see [15, Theorem 1.8 and Corollary 2.19], and therefore our original $\Lambda$ must be centrally generically tame.

Let us show that $G$ is algebraically bounded. From the last lemma (7.4), if $\mathbb{K}$ denotes the algebraic closure of $\mathbb{F}$, we know that $H^{\mathbb{K}} \cong U(\mathbb{K}\langle x,y\rangle)$ is again a pregeneric $\mathbb{K}\langle x,y\rangle$-module. So $H$ is an algebraically rigid pregeneric $\mathbb{F}\langle x,y\rangle$-module. Since $Z^{\mathbb{K}}$ is free of finite rank by the right, we obtain that $G^{\mathbb{K}} \cong Z^{\mathbb{K}} \otimes_{\mathbb{K}\langle x,y\rangle} H^{\mathbb{K}}$ is an endofinite $\Lambda^{\mathbb{K}}$-module. Then, $G^{\mathbb{K}} \cong \oplus_{i=1}^{n} \oplus_{I_i} G_i$ a possibly infinite direct sum of a finite number of pairwise non-isomorphic indecomposable endofinite $\Lambda^{\mathbb{K}}$-modules $G_1, \ldots, G_n$. Now, by [15, 1.8], $\Lambda^{\mathbb{K}}$ is generically tame, hence a tame algebra. This implies that each generic $G_i$ admits a structure of a $\Lambda^{\mathbb{K}}$-$\mathbb{K}(x)$-bimodule and $\mathrm{endol}\,(G_i) = \dim_{\mathbb{K}(x)}(G_i)$. Proceeding as in the beginning of the proof of [15, 3.2], we can produce a finite field extension $\mathbb{F}_i$ of $\mathbb{F}$ and a generic $\Lambda^{\mathbb{F}_i}$-module $\underline{G}_i$ such that $\underline{G}_i^{\mathbb{K}} \cong G_i$. Similarly, each finite-dimensional indecomposable $G_i$ has the form $G_i \cong \underline{G}_i^{\mathbb{K}}$, for some finite field extension $\mathbb{F}_i$ of $\mathbb{F}$ and some $\underline{G}_i \in \Lambda^{\mathbb{F}_i}$-mod. For notational simplicity, we can assume that all the fields $\mathbb{F}_1, \ldots, \mathbb{F}_n$ coincide and we denote them by $\mathbb{E}$. Again, since $Z^{\mathbb{E}} \otimes_{\mathbb{E}\langle x,y\rangle} -$ is sharp, $G^{\mathbb{E}} \cong Z^{\mathbb{E}} \otimes_{\mathbb{E}\langle x,y\rangle} H^{\mathbb{E}}$ is a generic $\Lambda^{\mathbb{E}}$-module. Moreover, we have that $G^{\mathbb{K}} \cong (G^{\mathbb{E}})^{\mathbb{K}}$ and $\underline{G}_i^{\mathbb{K}}$ share a direct summand, then by [14, 3.3], we obtain that $\underline{G}_i \cong G^{\mathbb{E}}$. Thus, $n = 1$, $G^{\mathbb{K}} \cong G_1$ is a generic module, and $G^{\mathbb{E}}$ is algebraically rigid. In particular, $G$ is algebraically bounded. □

**Acknowledgements.** The second author acknowledges the support of *Secretaría de Educación Pública* grant P/PIFI-2013-31MSU0098J-14.

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